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ε-neighborhoods of orbits of dynamical system

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Content

Research groups in Centre for Nonlinear Dynamics

- Low Dimensional Dynamics (Štimac, Anušić with Henk Bruin and Michal Misiurewicz)
- Fractal Zeta Functions (Žubrinić, Radunović with Michel Lapidus)
- Bifurcation Theory (Žubrinić, Županović, Resman, Horvat Dmitrović, Vlah with Pavao Mardešić)
- Nonuniform Hyperbolicity (Dragičević with Luis Barreira and Claudia Valls)
- Extended Dynamical Systems (Slijepčević, Rabar, Ninčević with Thierry Gallay)

Bifurcation Theory, Fractal Zeta Functions

Bifurcation Theory

Mardešić

Resman

Horvat

Vlah

Fractal Zeta Functions

Žubrinić

Radunović

Limit cycle

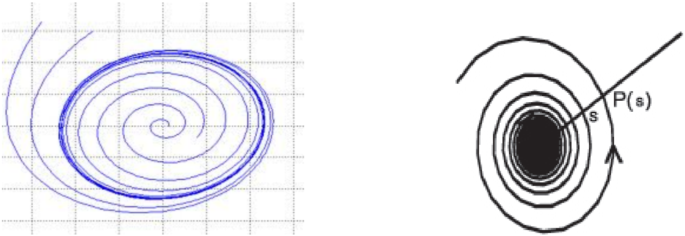


Figure: Limit cycle and Poincaré map

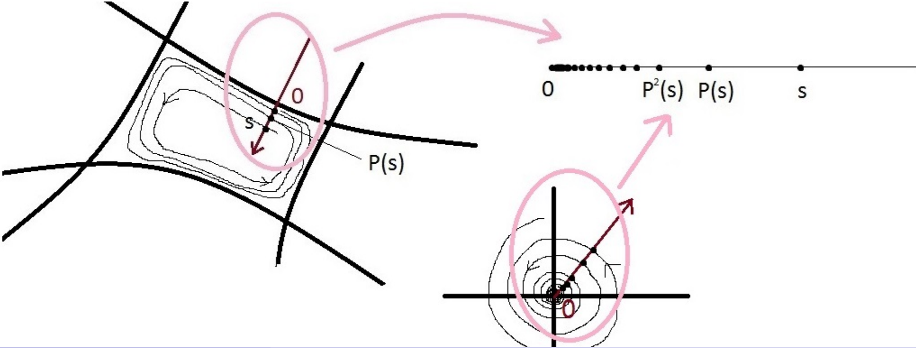
- *Limit cycles*, orbits which are isolated closed curves.
- Number of limit cycles $\Longleftrightarrow$ number of fixed points of the first return *Poincaré map* near singular point, limit cycle or polycycle.
- The first return near saddle polycycle of planar analytic vector is called *Dulac map*.

Why the Dulac maps and Poincaré maps are interesting in dynamics?

→ 16^{th} *Hilbert (open) problem*: number of closed periodic orbits born in bifurcations of a polynomial system?

→ number of closed periodic orbits is detected as **fixed points of the first return map** $s \to P(s)$

→ saddle polycycles - non-analytic Poincaré maps



Analytic and non-analytic Poincaré maps

- Poincaré map near weak focus is analytic (also near focus with no characteristic direction)
- Poincaré map near limit cycle is analytic
- Poincaré map (Dulac) near saddle polycycles is non-analytic
- Poincaré map near focus (general case) is non-analytic

Dulac maps

- The *Dulac maps*

1. germs f of $C^\infty(0,d)$ diffeomorphisms
2. with *power-log* asymptotic expansions[1] $\widehat{f}$:

$$f \sim \widehat{f},\ x \to 0,\ \text{with } \widehat{f}(x) = \sum_{i=1}^{\infty} x^{\alpha_i} P_i(-\log x),$$

P_i polynomials, $0 < \alpha_1 < \alpha_2 < \ldots \to +\infty$, finitely generated.

[1] $\forall n \in \mathbb{N},\quad \left| f - \sum_{i=1}^{n} a_i x^{\alpha_i} P_i(-\log x) \right| = O(x^{\alpha_n}),\ x \to 0.$

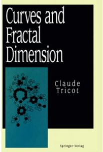
Dulac maps of hyperbolic polycycle

• the Dulac map of a monodromic saddle polycycle of planar analytic vector field

• a hyperbolic polycycle with k saddles of hyperbolicity ratios $r_1, \ldots, r_k > 0$:

$$s^{\alpha}(-\log s)^m, \;\; m \in \mathbb{N}, \;\; \alpha \text{ finitely generated by } r_1, \ldots, r_k$$

Definition of box dimension



ε-neighborhood of a bounded set $A \subset \mathbb{R}^n$

$$A_\varepsilon = \{y \in \mathbb{R}^n \,:\, d(y, A) < \varepsilon\}.$$

$$d = \dim_B A = n - \lim_{\varepsilon \to 0} \frac{\log |A_\varepsilon|}{\log \varepsilon}$$

which means that

$$|A_\varepsilon| \simeq \varepsilon^{n-d},$$

i.e. there exist positive constants C_1, C_2

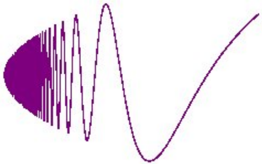
$$C_1 \varepsilon^{n-d} \leq |A_\varepsilon| \leq C_2 \varepsilon^{n-d}$$

for small ε.

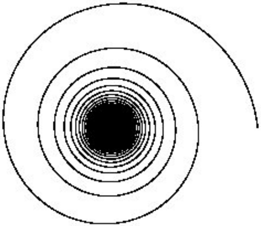
Examples appearing as orbits of dynamical systems

Chirp $f(x) = x^{\alpha} \sin x^{-\beta}$ $d = \dim_B \Gamma = 2 - \frac{1+\alpha}{1+\beta}$, $0 < \alpha < \beta \leq 1$

Spiral $r = \varphi^{-\alpha}$ $d = \dim_B \Gamma = \frac{2}{1+\alpha}$, $0 < \alpha \leq 1$



$\dim_B \Gamma = \frac{7}{4}$, $\alpha = 0.5, \beta = 1$



$\dim_B \Gamma = \frac{4}{3}$, $\alpha = 0.5$

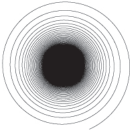
Sequence

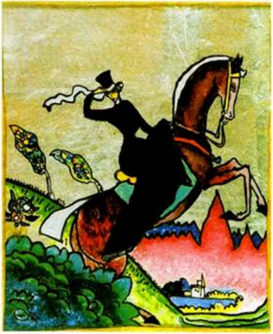
$S \ldots (\frac{1}{n^{\alpha}})$ $d = \dim_B S = \frac{1}{1+\alpha}$, $\alpha > 0$

Why box dimension of smooth curves?

"A straight line is the total negation of the plane whereas a curved line is potentially the plane in that it contains the essence of the plane within itself." Wassily Kandinsky: Point, Line and Plane, (1926)

Wassily Kandinsky (Moscou 1866- Neuilly-sur-Seine 1944)



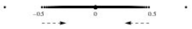


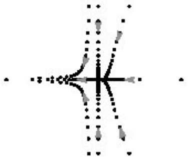
Motivation

- A natural idea is that "density" of orbit is related to quantity and quality of objects which could be produced by perturbation of the system.
- We are interested in connection between the change of box dimension and bifurcation of dynamical system.
- We are interested in connection between the value of box dimension of an orbit and multiplicity of the system near fixed point or periodic orbit.
- In general- we are interested in reading properties of dynamical system form epsilon-neighborhood of orbit

Which objects we study?

- Discrete systems by ε-neighborhood of an orbit near fixed point
  -normal forms of some classes of discrete systems could be read from ε-neighborhood (parabolic diffeos, Resman DCDS, 2013.)
  -discrete system could be embedded in a continuous system (class of 1-dim systems, Mardešić, Resman, Rolin, Ž., Advances in Math. 2016.) -instead of ε-neighborhood of discrete system we can study an ε-neighborhood of the continuous system
- Continuous systems by
  -spiral trajectories near focus, limit cycle and a polycycle
  -discrete system generated by Poincaré, Dulac map and unit-time map -embeddings of Poincaré, Dulac map and unit-time map





Methods

- We combine standard methods appearing in dynamical systems with $\varepsilon-$neighborhood approach
- Box dimension from leading term, and other exponents and coefficients show important properties of the system
- Standard methods: normal forms of discrete and continuous systems, formal and analytic classification by normal forms
- Asymptotic expansion of Poincaré map, Dulac maps, time one map
- Embeddings of discrete systems to continuous systems
- Complexification
- Oscillatory integrals, Abelian integrals,..
- Newton diagrams, blow-up, blow-up in a family of systems.....
- Numerical methods
- Slow-fast dynamics

Fractal dimensions in dynamics-standard approach

- Other fractal dimensions important for dynamics: Hausdorff dimension, Lyapunov dimension, Rényi spectrum for dimensions, correlation dimension, information dimension, Hentschel-Procaccia spectrum for dimensions, packing dimension, and effective fractal dimension.
- Since 1970 thermodynamics formalism, developed by Sinai, Ruelle, and Bowen, resulted in Hausdorff dimension of the Smale horseshoe and a results about Hausdorff dimension of Julia and Mandelbrot sets.
- Since 1980 physicists started to estimate and compute fractal dimensions of strange attractors (Lorenz, Henon,...). Fractal dimensions are estimated also for attractors of infinite-dimensional dynamical systems.

Theorem Poincaré map [Žubrinić, Ž, 2008]

Theorem

Assume: Γ *a spiral trajectory of a system of class* C^1; $P_\sigma(s)$ *is the Poincaré map with respect to an axis* σ, $P_\sigma(s) = s + d_\sigma(s)$ *for each* σ; *the displacement function* $d_\sigma(\cdot) : (0, r_\sigma) \to (-\infty, 0)$ *monotonically nonincreasing;* $-d_\sigma(s) \simeq s^\alpha$ *as* $s \to 0$, *for* $\alpha > 1$.

(a) If Γ *is a limit cycle spiral, then*

$$\dim_B \Gamma = 2 - \frac{1}{\alpha}.$$

(b) If Γ *is a focus spiral of (1), then*

$$\dim_B \Gamma = \begin{cases} 2 - \frac{2}{\alpha} & \text{for } \alpha > 2, \\ 1 & \text{for } 1 < \alpha \leq 2. \end{cases}$$

Weak focus

Weak focus is included in the previous theorem, since the Poincaré map is analytic

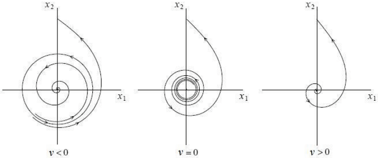
$$\begin{aligned} \dot{x} &= -y + p(x, y) \\ \dot{y} &= x + q(x, y), \end{aligned} \tag{1}$$

$p(x, y)$ and $q(x, y)$ are analytic functions and $|p(x, y)| \leq C(x^2 + y^2)$, $|q(x, y)| \leq C(x^2 + y^2)$ for some $C > 0$ and (x, y) near the origin.

Weak focus in the normal form and Hopf bifurcation [Žubrinić, Ž, 2005]

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} = 1. \end{cases} \tag{2}$$

Hopf bifurcation occurs for $l = 1$ if $a_0 = 0$. Hopf-Takens bifurcation occurs for $l > 1$, producing l limit cycles in the system



Theorem

Γ a part of a trajectory of (2) near the origin.
(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable with $r = e^{a_0\varphi}$, and hence $\dim_B \Gamma = 1$.
(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with the spiral $r = \varphi^{-1/2k}$, and

$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Discrete orbit

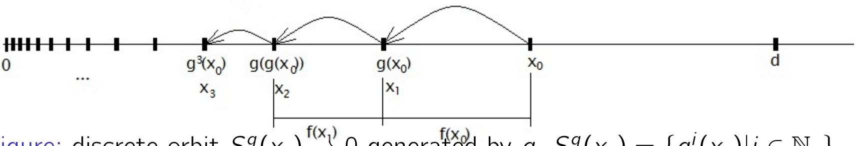


Figure: discrete orbit $S^g(x_0) \to 0$ generated by g, $S^g(x_0) = \{g^i(x_0) | i \in \mathbb{N}_0\}$.

generator g		**orbit** $S^g(x_0)$		
multiplicity of fixed point 0	$\leftrightarrow$	rate of growth of ε-neighbourhood	$\leftrightarrow$	*box dimension (g diff. at 0) *critical Minkowski order (g not diff. at 0)

$$g = \mathrm{id} - f$$

Box dimension and Minkowski content of $1-$dimensional set

- s-dimensional Minkowski content of the orbit $S^g(x_0)$, $0 \leq s \leq 1$:
$\mathcal{M}^s(S^g(x_0)) = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon(S^g(x_0))|}{\varepsilon^{1-s}}$
- box dimension $s = \dim_B S^g(x_0)$
- $\mathcal{M}^s \neq 0, \infty \Rightarrow |A_\varepsilon(S^g(x_0))| \simeq \varepsilon^{1-s}$, otherwise not comparable to any power of ε

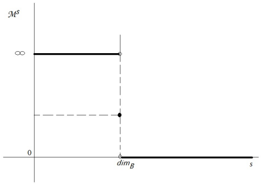


Figure: Minkowski content $\mathcal{M}^s$ as function of $s \in [0, 1]$

A motivating example

EXAMPLE 1.

- $g_1(x) = x - x^2$, (diff. generators)
- $g_2(x) = x - x^2(-\log x)$, $g_3(x) = x - x^2 \log(-\log x)$ (nondiff. generators)
- $|A_\varepsilon(S^{g_1}(x_0))| \simeq \varepsilon^{1/2}$ – power-type behaviour!
- $\lim_{\varepsilon\to 0} \frac{|A_\varepsilon(S^{g_{2,3}})|}{\varepsilon^{1/2}} = +\infty$, $\lim_{\varepsilon\to 0} \frac{|A_\varepsilon(S^{g_{2,3}})|}{\varepsilon^{1/(2+\delta)}} = 0$, $\forall\delta > 0$ – noncomparable to any power!
- $\dim_B S^{g_1}(x_0) = \frac{1}{2}$, but also $\dim_B S^{g_{2,3}}(x_0) = \frac{1}{2}$
- Minkowski content

$$\mathcal{M}^{1/2}(S^{g_1}(x_0)) > 0, \text{ but both } \mathcal{M}^{1/2}(S^{g_{2,3}}(x_0)) = 0$$

- In nondiff. case find appropriate gauge functions (instead of powers) to compare $|A_\varepsilon|$ with $\to$ generalized Minkowski content (Lapidus)

The behaviour of the ε-neighbourhood of the orbit with respect to nondifferentiable generator

Theorem (Mardešić, Resman, Županović, 2012)

- *$f \in C^r(0,d)$, continuous on $[0,d)$, positive on $(0,d)$,* $f(0) = f'(0) = 0$,
- *f sublinear:*

$$m \leq x \cdot (\log f)'(x),\ \ x \in (0,d),\ \ m > 1.$$

Then

$$|A_\varepsilon(S^g(x_0))| \simeq f^{-1}(\varepsilon) \text{ as } \varepsilon \to 0.$$

$\star$ e.g. $f(x) = \frac{x}{-\log x}$ not sublinear, $\frac{|A_\varepsilon(S^g(x_0))|}{f^{-1}(\varepsilon)} \to \infty$ as $\varepsilon \to 0$.

Special case-differentiable generator

Corollary

f enough differentiable on $[0, d)$, positive, strictly increasing on $(0, d)$, $f(x) \simeq x^k$, $g = id - f$ then $|A_\varepsilon(S^g(x_0))| \simeq \varepsilon^{1/k}$ and $\dim_B(S^g(x_0)) = 1 - \frac{1}{k}$

Our admissible class of generating functions

- f with asymptotic development in Chebyshev scale at $x = 0$,

Definition (CHEBYSHEV SCALE;
Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

$\mathcal{I} = \{u_0, u_1, u_2, \ldots\}$, $u_i \in C[0,d) \cap C^r(0,d)$, $r \in \mathbb{N} \cup \{\infty\}$ such that

i) Division/differentiation algorithm can be performed r times except possibly at $x = 0$ ($\Rightarrow$ extension by continuity to 0):

$$\mathcal{I} = \{u_0, u_1, u_2, \ldots\} /{:}\, u_0 \Rightarrow D_0(\mathcal{I}) = \{\underbrace{1}_{D_0(u_0)}, \underbrace{\frac{u_1}{u_0}}_{D_0(u_1)}, \underbrace{\frac{u_2}{u_0}}_{D_0(u_2)}, \ldots\} \,/()'$$

$$\{D_0(u_1))', D_0(u_2)', \ldots\} /{:} D_0(u_1)' \Rightarrow D_1(\mathcal{I}) = \{\underbrace{1}_{D_1(u_1)}, \underbrace{\frac{(D_0(u_2))'}{(D_0(u_1))'}}_{D_1(u_2)}, \underbrace{\frac{(D_0(u_3))'}{(D_0(u_0))'}}_{D_1(u_3)}, \ldots\} /()'$$

$$\{D_1(u_2))', D_1(u_3)', \ldots\} /{:} D_1(u_2)' \Rightarrow D_2(\mathcal{I}) = \{\underbrace{1}_{D_2(u_2)}, \underbrace{\frac{(D_1(u_3))'}{(D_1(u_2))'}}_{D_2(u_3)}, \underbrace{\frac{(D_1(u_4))'}{(D_1(u_1))'}}_{D_2(u_4)}, \ldots\} /()'$$

ii) $D_i(u_{i+1})$ strictly increasing

iii) $D_j u_i(0) = 0$, $j < i$ in the sense of limit

$D_i(f) \ldots$ *$i-$th generalized derivative of f in the scale $\mathcal{I}$*

Examples of Chebyshev scales

- $\mathcal{I} = \{1, x, x^2, x^3, x^4, \ldots\}$ -diff. case,
- $\mathcal{I} = \{x^{\alpha_0}, x^{\alpha_1}, x^{\alpha_2}, \ldots\}$, $\alpha_i \in \mathbb{R}$, $0 < \alpha_0 < \alpha_1 < \alpha_2 < \ldots$
- $\mathcal{I} = \{1, x(-\log x), x, x^2(-\log x), x^2, x^3(-\log x), x^3, \ldots\}$
- any set of monomials of the type $x^k(-\log x)^l$, ordered by increasing flatness:
- $x^i(-\log x)^j < x^k(-\log x)^l$ if and only if $(i < k)$ or $(i = j$ and $j > l)$.

Generalized Minkowski content, critical Minkowski order (generalization of box dimension)

- $\mathcal{I} = \{u_0, u_1, \ldots\}$ Chebyshev, $u_{i,\ i>0}$ positive, strictly increasing on $(0, d)$, f has development in $\mathcal{I}$
- assumptions from Theorem 1 on f, and the upper power condition

Definition (generalized Minkowski content)

$*$ *Upper generalized Minkowski content of $S^g(x_0)$ with respect to a Chebyshev scale $\{u_i,\ i = 1, 2, \ldots\}$*

$$\mathcal{M}^*(S^g(x_0), u_i) = \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon(S^g(x_0))|}{u_i^{-1}(\varepsilon)}$$

-
- $|A_\varepsilon(S^g(x_0))|$ compared to inverted T-scale, not to powers of ε

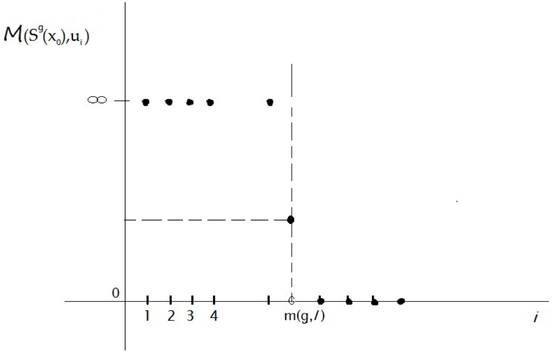


Figure: upper generalized Minkowski content as function of i.

Definition (critical Minkowski order)

$*$ *Upper critical order of g with respect to the scale $\mathcal{I}$:*

$$\overline{m}(g,\mathcal{I}) \;=\; \max\{i \geq 1 \mid \mathcal{M}^*(S^g(x_0), u_i) > 0\},$$

$*$ *(lower) critical order $\underline{m}(g,\mathcal{I})$, $m(g,\mathcal{I})$*

$*$ $m(g,\mathcal{I}) = i_0$ iff $|A_\varepsilon(S^g(x_0))| \simeq u_{i_0}^{-1}$

-g differentiable at zero:
development in $\mathcal{I} = \{1, x, x^2, \ldots\} \Rightarrow \dim_B(g) = 1 - \frac{1}{m(g,\mathcal{I})}$.

Multiplicity of fixed point zero of g-differentiable case and in a family

$g \in C^r[0, d)$, 0 fixed point; f =id$-g$

- $\mu_0(f) = k$, if $f(0) = f'(0) = \ldots = f^{(k-1)}(0) = 0$, $f^{(k)}(0) \neq 0$
- $\mu_0^{fix}(g) := \mu_0(f) = k$

- $g, (g_\lambda)$ family
- $\mu_0(g, (g_\lambda)) \geq m$ if for any neighbourhood of $x = 0$ there exists some function in (g_λ), arbitrarly close to g, with at least m fixed points in the given neighbourhood (different from 0)
- standard multiplicity in diff. case = multiplicity of f in a family of all diff. functions
- (Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

Connection multiplicity - critical Minkowski order

- (f_λ), asymptotic development in a family of T-scales $\mathcal{I}_\lambda$

Theorem (MRŽ, 2012)

$f = f_{\lambda_0}$ satisfies all assumptions of Theorem 2 and the upper power condition:

$$x \cdot (\log f(x))' \leq M, \ x \in (0, d), \ \textit{for some constant } M > 0.$$

Then the following is equivalent:

1. $D_i(f)(0) = 0$ *for* $i = 0, \ldots, k-1$ *and* $D_k(f)(0) > 0, \ k \geq 1$, $(f \simeq u_k, \ k \geq 1)$,
2. $|A_\varepsilon(S^g(x_0)| \simeq u_k^{-1}(\varepsilon)$,
3. $\mu_0^{fix}(g, (g_\lambda)) = k$,
4. $m(g, \mathcal{I}) = k$.

Deficiency of box dimension, nondifferentiable generators

EXAMPLE 1 REVISITED

- f_2, f_3 not differentiable at $x = 0$ (not of power-type behaviour as $x \to 0$)
- standard box dimension/Minkowski contents compare $|A_\varepsilon(S^{g_{2,3}}(x_1))|$ to power functions; $|A_\varepsilon(S^{g_{2,3}}(x_1))| \simeq f_{2,3}^{-1}(\varepsilon)$ not of power type $\Rightarrow$ no precise information on behaviour of ε-neighbourhood
- critical Minkowski order with respect to the scale

$$\mathcal{I} = \{1, x^2 \log(-\log x), x^2(-\log x), x^2, \ldots\} :$$

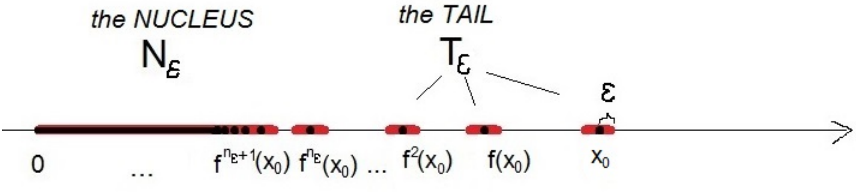
- $m(g_1, \mathcal{I}) = 3 > m(g_2, \mathcal{I}) = 2 > m(g_3, \mathcal{I}) = 1.$

Applications of results

- To find multiplicity of differentiable and nondifferentiable Poincaré maps around different limit periodic sets weak/strong focus, limit cycle, saddle loop, 2 saddle loop, also to obtain multiplicity of the Abelian integrals

Roussarie's expansion of Poincaré map around saddle loop: logarithmic terms $x^k(-\log x) \rightarrow$ nondiff. at $x = 0$.

Nucleus and tail



- $\boxed{\varepsilon \mapsto \ A_\varepsilon(S^f(x_0)) = \ell(N_\varepsilon) + \ell(T_\varepsilon) = (f^{n_\varepsilon}(x_0) + 2\varepsilon) + n_\varepsilon \cdot 2\varepsilon}$, $\varepsilon > 0$

(*C. Tricot*)

An improvement of the definition of the length of the ε-neighborhood, [MRRZ2] Mardešić, Resman, Rolin, V. Ž., Length of ε-neighborhoods of orbits of Dulac maps, preprint

Suppose $g \in C^\infty(0,d)$ embeddable into a (C^1 in t) flow[1],

$g \hookrightarrow \{g^t\}_{t\in\mathbb{R}}$, $g^t \in C^\infty(0,d)$, as the time-one map ($g = g^1$).

• the **continuous critical time** τ_ε: $g^{\tau_\varepsilon}(x_0) - g^{\tau_\varepsilon+1}(x_0) = 2\varepsilon$

$\Rightarrow n_\varepsilon = \lceil \tau_\varepsilon \rceil$ • The **continuous-time length of ε-neighborhood of orbit**:

$$\boxed{A_\varepsilon^c(S^g(x_0)) = g^{\tau_\varepsilon}(x_0) + 2\varepsilon + \tau_\varepsilon \cdot 2\varepsilon \quad \big(\text{ the given flow } \{g_t\}_t\big)}$$

• We show that $\varepsilon \in (0,d)$ $\varepsilon \mapsto A_\varepsilon^c(S^g(x_0)))$ in ε has a full asymptotic expansion in the power, iterated logarithm scale

• The asymptotic expansion extends the initial part of the asymptotic expansion of the classical $A_\varepsilon(S^g(x_0))$

[1] $g^0 = \mathrm{id}$, $g^{t+s} = g^t \circ g^s$; $t, s \in \mathbb{R}$.

An example from [MRRZ2]

- $g(x) = x + x^2 \log x + \dots$ Dulac.
- $$|A_\varepsilon^c(S^g(x_0)| = \sqrt{2}\varepsilon^{1/2}\boldsymbol{\ell}^{1/2} + \dots$$

has asymptotic expansion in $\mathcal{L}_2$, $\varepsilon \to 0$ $\boldsymbol{\ell} := \boldsymbol{\ell}_1 := \frac{1}{-\log x}$, $\mathcal{L}_2$ algebra with $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2 = \boldsymbol{\ell} \circ \boldsymbol{\ell}$

... a refinement of a previous result [MRZ, 2012]:

$$|A_\varepsilon(S^g(x_0)| \simeq f^{-1}(\varepsilon),\ f = \mathrm{id} - g$$

Here, $f(x) = \mathrm{id} - g = x^2(-\log x) + \dots$

$$\Rightarrow \mathrm{f}^{-1}(y) \sim \frac{\sqrt{2}y^{1/2}}{(-\log y)^{1/2}}.$$

Thank you!

Asymptotic development of displacement functions

(X_λ) analytic unfolding of (X_{λ_0})(monodromic LPS), (f_λ)

- stable strong/weak focus singular point

$$f_\lambda(x) = \beta_1(\lambda)(x+g_1(\lambda,x))+\beta_3(\lambda)(x^3+g_3(\lambda,x))+\beta_5(\lambda)(x^5+g_5(\lambda,x))+\ldots,\ x\in[0,d),$$

$g_i(\lambda,x)$ linear combination of power-type monomials of order strictly greater than x^i, coefficients depending on λ.

⋆ strong focus: $\beta_1(\lambda_0)\neq 0$, weak focus: $\beta_1(\lambda_0)=0$.

- stable limit cycle

$$f_\lambda(x) = \alpha_0(\lambda)+\alpha_1(\lambda)x+\alpha_2(\lambda)x^2+\alpha_3(\lambda)x^3+\ldots,\ x\in[0,d).$$

- stable homoclinic loop

$$\begin{aligned} f_\lambda(x) &= \beta_0(\lambda)+\alpha_1(\lambda)[x\omega(x,\alpha_1(\lambda))+g_1(x,\lambda)]+\\ &+ \beta_1(\lambda)x+\alpha_2(\lambda)[x^2\omega(x,\alpha_1(\lambda))+g_2(x,\lambda)]+\beta_2(\lambda)x^2+\ldots+\\ &+ \beta_n(\lambda)x+\alpha_n(\lambda)[x^n\omega(x,\alpha_1(\lambda))+g_n(x,\lambda)]+\beta_n(\lambda)x^n+o(x^n),\end{aligned}$$

$$\omega(x,\alpha)=\begin{cases}\frac{x^{-\alpha}-1}{\alpha} & \text{if } \alpha\neq 0,\\ -\log x & \text{if } \alpha=0,\end{cases}\qquad x\in(0,d),$$

$g_i(x,\lambda)$ linear combination of monomials of the type $x^k\omega^l$ of strictly greater order than $x^i\omega$: $x^i\omega^j < x^k\omega^l$ if $(i<k)$ or $(i=k$ and $j>l)$.
⋆ $\alpha_1(\lambda_0)=0,\ \beta_0(\lambda_0)=0$.

The example: cyclicity of the stable homoclinic loop via the critical order

The corresponding family of Chebyshev scales:

$$\mathcal{I}_\lambda = \{1, x\omega(x,\alpha_1(\lambda)) + g_1(x,\lambda), x, x^2\omega(x,\alpha_1(\lambda)) + g_2(x,\lambda), x^2, \ldots\}.$$

The development of f_{λ_0} around stable loop:

$$\begin{aligned} f_{\lambda_0}(x) &= \beta_1(\lambda_0)x + \alpha_2(\lambda_0)x^2\omega(x,0) + \alpha_3(\lambda_0)x^3\omega(x,0) + \ldots = \\ &= \beta_1(\lambda_0)x + \alpha_2(\lambda_0)x^2(-\log x) + \alpha_3(\lambda_0)x^3(-\log x) + \ldots (3) \end{aligned}$$

- If $f_{\lambda_0}(x) \simeq x^k$ as $x \to 0$, $k \geq 2$, then $m(g_{\lambda_0}, \mathcal{I}_{\lambda_0}) = 2k$.
- If $f_{\lambda_0} \simeq x^k(-\log x)$, $k \geq 2$, then $m(g_{\lambda_0}, \mathcal{I}_{\lambda_0}) = 2k-1$.
- The cyclicity of the loop less than or equal to $2k$, $2k-1$; critical order recognizes cyclicity!
- $\dim_B(S^{g_{\lambda_0}}(x_0))$ in both cases $1-1/k$; box dimension does not recognize cyclicity!

Multiplicity of a function in a family

Definition

Λ *a topological space,* $\{f_\lambda | \lambda \in \Lambda\}$, $f_\lambda : [0, d) \to \mathbb{R}$

- $x = 0$ *is* **a zero of multiplicity greater than or equal to** m **of the function** f_{λ_0} **in the family of functions** (f_λ) *if there exists a sequence of parameters* $\lambda_n \to \lambda_0$ *as* $n \to \infty$ *such that for every* $n \in \mathbb{N}$, f_{λ_n} *has* m *distinct zeros* $y_1^n, \dots, y_m^n \in [0, d)$ *different from* $x = 0$ *and* $y_j^n \to 0$, *as* $n \to \infty$, $j = 1, \dots, m$.
- *If* m *is the biggest possible such,* $x = 0$ *is* **a zero of of multiplicity** m **of the function** f_{λ_0} **in the family** (f_λ), $\mu_0(f_{\lambda_0}, (f_\lambda)) = m$.
- $g_\lambda = id - f_\lambda$, *the multiplicity of* 0 *as a fixed point of* g_{λ_0} *with respect to the family* (g_λ) *is* $\mu_0^{fix}(g_{\lambda_0}, (g_\lambda)) := \mu_0(f_{\lambda_0}, (f_\lambda))$.

Admissible class, asymptotic development in a family of Chebyshev scales

Admissible class of generators

- $f = f_{\lambda_0}$ belongs to the family (f_λ) of functions on $[0, d)$ with asymptotic development in a Chebishev family $\mathcal{I}_\lambda = \{u_0(x, \lambda), u_1(x, \lambda), \ldots, u_k(x, \lambda)\}$:

$$f_\lambda(x) = \sum_{i=0}^{k} \alpha_i(\lambda) u_i(x, \lambda) + \psi_k(x, \lambda), \quad \lambda \in \mathbf{P},$$

  $\psi_k(x, \lambda)$ such that all generalized derivatives up to the order k vanish in $x = 0$.

- $g_\lambda = id - f_\lambda$, $\lambda \in \mathbf{P}$; $g = g_{\lambda_0}$.

⋆ the upper power condition:

$$\mathcal{I} = \{e^{-1/x}, e^{-1/2x}, e^{-1/3x}, \dots\},$$

then $u_1^{-1}(\varepsilon) \simeq u_2^{-1}(\varepsilon) \simeq u_3^{-1}(\varepsilon) \simeq \dots$

The multiplicity **cannot be uniquely determined from the behaviour of** $\varepsilon-$**neighbourhood!**

⋆ **e.g.** $f(x) = \frac{x}{-\log x}$ **not sublinear,** $\frac{|A_\varepsilon(S^g(x_0))|}{f^{-1}(\varepsilon)} \to \infty$ **as** $\varepsilon \to 0$**.**

non-Chebyshev: $\mathcal{I} = \{1, x^2, x^2/2, x^2/3, \dots\}$ **(!(ii),(iii)!)**

non-Chebyshev: $\mathcal{I} = \{1, x \sin\frac{1}{x}, x^2 \sin\frac{1}{x}, \dots\}$ **(!(i)!)**