Invariant surfaces and first integrals of the May-Leonard asymmetric system

Valery Romanovski

Faculty of Electrical Engineering and Computer Science,
University of Maribor, Slovenia, Maribor, Slovenia
CAMTP – Center for Applied Mathematics and Theoretical Physics,
University of Maribor, Slovenia
Faculty of Natural Science and Mathematics, University of Maribor,
Maribor, Slovenia

Vipava, December 8, 2017

References:

- V. Antonov, D. Dolićanin, V. G. Romanovski, J. Tóth, Invariant planes and periodic oscillations in the May-Leonard asymmetric model, MATCH Commun. Math. Comput. Chem. 2016, vol. 76, no.2, 455–474.
- V. Antonov, W. Fernandes, V.G. Romanovski, N.L. Shcheglova, First integrals of the May-Leonard asymmetric system, 2016, submitted.

Table of contents

Predator-prey model

Consider a biological system in which two species interact, one a predator and one its prey. They evolve in time according to the pair of the equations:

$$\frac{dx}{dt} = x(\alpha - \beta y), \frac{dy}{dt} = -y(\gamma - \delta x) \tag{1}$$

where,
y is the number of some predator;
x is the number of its prey;
$\frac{dx}{dt} = \dot{x}$ and $\frac{dy}{dt} = \dot{y}$ represent the growth of the two populations against time t.

- System (1) is called Lotka-Volterra system

The prey equation:

$$\frac{dx}{dt} = \alpha x - \beta xy.$$

The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented by the term αx. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented by βxy.

The predator equation:

$$\frac{dy}{dt} = \delta xy - \gamma y.$$

δxy - the growth of the predator population. γy represents the loss rate of the predators due to either natural death or emigration; it leads to an exponential decay in the absence of prey.

The equation expresses the change in the predator population as growth fuelled by the food supply, minus natural death.

May-Leonard system

Mathematical model introduced by May and Leonard in 1975 to describe a competition of three species:

$$\begin{aligned} \dot{x} =& x(1 - x - \alpha y - \beta z), \\ \dot{y} =& y(1 - \beta x - y - \alpha z), \\ \dot{z} =& z(1 - \alpha x - \beta y - z). \end{aligned} \tag{2}$$

$x, y, z \geq 0$, $0 < \alpha < 1 < \beta$, and $\alpha + \beta > 2$.

Some studies about its dynamics:

- May and Leonard (1975), dynamic aspects;
- Schuster, Sigmund and Wolf (1979), dynamic aspects;
- Leach and Miritzis (2006), first integrals;
- Blé, Castellanos, Llibre and Quilantán (2013), integrability.

A generalization of (2):

$$\begin{aligned}\dot{x} =& x(1 - x - \alpha_1 y - \beta_1 z) = X(x, y, z),\\ \dot{y} =& y(1 - \beta_2 x - y - \alpha_2 z) = Y(x, y, z),\\ \dot{z} =& z(1 - \alpha_3 x - \beta_3 y - z) = Z(x, y, z),\end{aligned} \tag{3}$$

$x, y, z \geq 0$ and α_i, β_i, $(1 \leq i \leq 3)$ real parameters.

A generalization of (2):

$$\begin{aligned} \dot{x} =& x(1 - x - \alpha_1 y - \beta_1 z) = X(x, y, z), \\ \dot{y} =& y(1 - \beta_2 x - y - \alpha_2 z) = Y(x, y, z), \\ \dot{z} =& z(1 - \alpha_3 x - \beta_3 y - z) = Z(x, y, z), \end{aligned} \tag{3}$$

$x, y, z \geq 0$ and α_i, β_i, $(1 \leq i \leq 3)$ real parameters.

It is called the **May-Leonard asymmetric** model.

Some studies about its dynamics:

- Chi, Hsu and Wu (1998), dynamic aspects;
- van der Hoff, Greeff and Fay (2009), dynamic aspects;
- Antonov, Dolićanin, R. and Tóth (2016), dynamic aspects, first integrals.

Antonov, Dolićanin, V. R. and Tóth (2016):

- found first integrals of Darboux type constructed using invariant planes (Darboux polynomials of *degree one*);
- showed that the system can have a family of periodic solutions satisfying Lyapunov's theorem on holomorphic integral.

Existence of periodic solutions for May-Leonard asymmetric system was shown in [Chi, Hsu and Wu 1998], however it was mentioned there that the periodic solutions appear due to Hopf bifurcations.

Existence of periodic solutions for May-Leonard asymmetric system was shown in [Chi, Hsu and Wu 1998], however it was mentioned there that the periodic solutions appear due to Hopf bifurcations.

We show that there is a family of periodic solutions as follows.

Existence of periodic solutions for May-Leonard asymmetric system was shown in [Chi, Hsu and Wu 1998], however it was mentioned there that the periodic solutions appear due to Hopf bifurcations.

We show that there is a family of periodic solutions as follows.

When $\beta_3 = \frac{2-\alpha_1-\alpha_2+\alpha_1\alpha_2-\alpha_3+\alpha_1\alpha_3+\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_3-\beta_1-\beta_2+\beta_1\beta_2}{(\beta_1-1)(\beta_2-1)}$, the system has the invariant plane

$$H_4 = -x+\alpha_3 x+\beta_2 x-\alpha_3\beta_2 x+y-\alpha_1 y-\alpha_3 y+\alpha_1\alpha_3 y+z-\beta_1 z-\beta_2 z+\beta_1\beta_2 z \tag{4}$$

Existence of periodic solutions for May-Leonard asymmetric system was shown in [Chi, Hsu and Wu 1998], however it was mentioned there that the periodic solutions appear due to Hopf bifurcations.
We show that there is a family of periodic solutions as follows.
When
$\beta_3 = \frac{2-\alpha_1-\alpha_2+\alpha_1\alpha_2-\alpha_3+\alpha_1\alpha_3+\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_3-\beta_1-\beta_2+\beta_1\beta_2}{(\beta_1-1)(\beta_2-1)}$, the system has the invariant plane

$$H_4 = -x+\alpha_3 x+\beta_2 x-\alpha_3\beta_2 x+y-\alpha_1 y-\alpha_3 y+\alpha_1\alpha_3 y+z-\beta_1 z-\beta_2 z+\beta_1\beta_2 z \tag{4}$$

First integral:

$$\Psi = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} H_4^{\alpha_4} \tag{5}$$

$$\alpha_2 = -\frac{\alpha_1(-1+\beta_1)}{\alpha_2-1},\ \ \alpha_3 = \frac{\alpha_1(-1+\beta_1)(-1+\beta_2)}{(-1+\alpha_2)(-1+\alpha_3)},$$
$$\alpha_4 = -\frac{\alpha_1(1-\alpha_2+\alpha_2\alpha_3-\alpha_3\beta_1-\beta_2+\beta_1\beta_2)}{(-1+\alpha_2)(-1+\alpha_3)}.$$

To guarantee that the all coordinates of the singular point are positive we take the parameters
$\beta_1 = 1/4,\ \beta_2 = 11/10,\ \alpha_1 = 5/4,\ \alpha_2 = 4/5,\ \alpha_3 = 3/2, \beta_3 = 2/3.$
In this case system (3)

$$\dot{x} = x(-x - \frac{5y}{4} - \frac{z}{4} + 1),\ \dot{y} = y(-\frac{11x}{10} - y - \frac{4z}{5} + 1),\ \dot{z} = z(\frac{3x}{2} + \frac{2y}{3} + z - 1). \tag{6}$$

and the singular point P has the coordinates

$$x_0 = 1/3,\ y_0 = 1/2,\ z_0 = 1/6.$$

Proposition

System (6) has a family of periodic solutions in a neighborhood of the singular point $P(1/3, 1/2, 1/6)$.

Proof:

Moving the origin to the singular point by the substitution

$$u = x - x_0, v = y - y_0, \; w = z - z_0$$

and then performing the linear change of coordinates

$$\begin{aligned} u =& 2X + 370Y/249, \\ v =& 3X - Y - 15\sqrt{10}Z/83, \\ w =& X + 1/249(-235Y + 77\sqrt{10}Z) \end{aligned}$$

we obtain from (6)

$$\dot{X} = -X - 6X^2 + \frac{10450Y^2}{268671} + \frac{38048\sqrt{10}YZ}{806013} - \frac{10450Z^2}{268671},$$

$$\dot{Y} = \frac{Z}{3\sqrt{10}} - 6XY + \sqrt{\frac{2}{5}}XZ - \frac{2090Y^2}{39923} + \frac{16979\sqrt{\frac{2}{5}}YZ}{39923} + \frac{2090Z^2}{39923},$$

$$\dot{Z} = -\frac{Y}{3\sqrt{10}} - \sqrt{\frac{2}{5}}XY - 6XZ + \frac{19187\sqrt{10}Y^2}{119769} + \frac{7730YZ}{119769} - \frac{19187\sqrt{10}Z^2}{119769}.$$

- By the Center Manifold Theorem $\exists$ an analytic center manifold $X = h(X, Y)$ passing through $X = Y = Z = 0$.
- Expanding the first integral (5) into power series

$$\Psi(X, Y, Z) = Y^2 + Z^2 + h.o.t.$$

- $\Rightarrow$ in a neighborhood of the origin there exists a family of periodic orbits formed by the intersection of the graphs of $X = h(Y, Z)$ and $\Psi = c$ $(0 < c < c_0)$.

First integrals

Consider system of differential equations

$$\begin{aligned} \dot{x} &= P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= R(x, y, z), \end{aligned} \tag{7}$$

P, Q and R polynomials of degree at most m.

First integrals

Consider system of differential equations

$$\begin{aligned} \dot{x} =& P(x,y,z), \\ \dot{y} =& Q(x,y,z), \\ \dot{z} =& R(x,y,z), \end{aligned} \tag{7}$$

P, Q and R polynomials of degree at most m.

Let $\mathfrak{X}$ be the corresponding vector field,

$$\mathfrak{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

A C^1 function $H : U \to \mathbb{R}$ non-constant in any subset of U is a **first integral** of the differential system (7) if

$$\mathfrak{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} \equiv 0 \text{ in } U.$$

A C^1 function $H : U \to \mathbb{R}$ non-constant in any subset of U is a **first integral** of the differential system (7) if

$$\mathfrak{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} \equiv 0 \text{ in } U.$$

If there are two first integrals of system (7),

$$H_1 : U_1 \to \mathbb{R} \text{ and } H_2 : U_2 \to \mathbb{R}$$

we say that:

A C^1 function $H : U \to \mathbb{R}$ non-constant in any subset of U is a **first integral** of the differential system (7) if

$$\mathfrak{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} \equiv 0 \text{ in } U.$$

If there are two first integrals of system (7),

$$H_1 : U_1 \to \mathbb{R} \text{ and } H_2 : U_2 \to \mathbb{R}$$

we say that: H_1 and H_2 are **independent in** $U_1 \cap U_2$ if their gradients are independent in $U_1 \cap U_2$ (except perhaps in a zero Lebesgue measure set).

A C^1 function $H : U \to \mathbb{R}$ non-constant in any subset of U is a **first integral** of the differential system (7) if

$$\mathfrak{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} \equiv 0 \text{ in } U.$$

If there are two first integrals of system (7),

$$H_1 : U_1 \to \mathbb{R} \text{ and } H_2 : U_2 \to \mathbb{R}$$

we say that: H_1 and H_2 are **independent in** $U_1 \cap U_2$ if their gradients are independent in $U_1 \cap U_2$ (except perhaps in a zero Lebesgue measure set).

System (7) is **completely integrable** in $U \subset \mathbb{R}^3$ if it has two independent first integrals in U.

Darboux first integrals

- A **Darboux polynomial** of system (7)

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z),$$

is a polynomial $f(x,y,z)$ such that

$$\mathfrak{X} f = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q + \frac{\partial f}{\partial z} R = Kf, \tag{8}$$

where $K(x,y,z)$ is a polynomial of degree at most $m-1$,

Darboux first integrals

- A **Darboux polynomial** of system (7)

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z),$$

is a polynomial $f(x,y,z)$ such that

$$\mathfrak{X} f = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q + \frac{\partial f}{\partial z} R = K f, \tag{8}$$

where $K(x,y,z)$ is a polynomial of degree at most $m-1$, called **cofactor** of f.

Darboux first integrals

- A **Darboux polynomial** of system (7)

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z),$$

  is a polynomial $f(x,y,z)$ such that

$$\mathfrak{X} f = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q + \frac{\partial f}{\partial z} R = K f, \qquad (8)$$

  where $K(x,y,z)$ is a polynomial of degree at most $m-1$, called **cofactor** of f.

- If f is a Darboux polynomial of (7) then the equation $f=0$ defines an algebraic surface which is invariant under the flow of system (7).

If system (7) has irreducible invariant surfaces $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \lambda_i K_i = 0,$$

then

If system (7) has irreducible invariant surfaces $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \lambda_i K_i = 0,$$

then

$$H = f_1^{\lambda_1} \cdots f_k^{\lambda_k},$$

is a first integral of (7), called a **Darboux first integral**.

Elimination theory

Let I be an ideal in a polynomial ring $k[x_1, \ldots, x_n]$, where k is a field.

Elimination theory

Let I be an ideal in a polynomial ring $k[x_1, \ldots, x_n]$, where k is a field.

And ℓ be a fixed number from the set $\{0, 1, \ldots, n-1\}$.

Elimination theory

Let I be an ideal in a polynomial ring $k[x_1, \ldots, x_n]$, where k is a field.

And ℓ be a fixed number from the set $\{0, 1, \ldots, n-1\}$.

The ℓ-**th elimination ideal** of I is the ideal

$$I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n].$$

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$.

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \dots, x_n]$ with $x_1 > x_2 > \cdots > x_n$. Let G be a Gröbner basis for the ideal I with respect to this order.

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$. Let G be a Gröbner basis for the ideal I with respect to this order. Then for any $0 \leqslant \ell \leqslant n-1$, the set

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$. Let G be a Gröbner basis for the ideal I with respect to this order. Then for any $0 \leqslant \ell \leqslant n-1$, the set

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis for the ℓth elimination ideal I_ℓ.

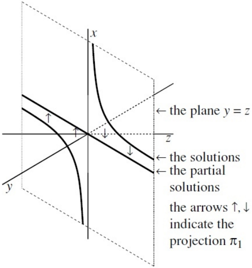
Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$. Let G be a Gröbner basis for the ideal I with respect to this order. Then for any $0 \leqslant \ell \leqslant n-1$, the set

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis for the ℓth elimination ideal I_ℓ.

Elimination – projection of the variety on the subspace $x_{\ell+1}, \dots, x_n$.
The variety of $\mathbf{V}(I_\ell)$ is the Zariski closure of the projection of $\mathbf{V}(I)$.
It is not always possible to extend a partial solution to a solution of the original system.
Example. $xy = 1, \quad xz = 1$. The reduced Groebner basis of $I = \langle xy - 1, xz - 1\rangle$ with respect to lex with $x > y > z$ is $\{xz - 1, y - z\}$.
$I_1 = \langle y - z\rangle$. $\mathbf{V}(I_1)$ is the line $y = z$ in the (y, z)-plane. Partial solutions corresponding to I_1 are $\{(a, a) : a \in \mathbb{C}\}$. Any partial solution (a, a) for which $a \neq 0$ can be extended to the solution $(1/a, a, a)$, except of $(0, 0)$.



Invariant surfaces of the May-Leonard asymmetric system

Objective

Find conditions on the parameters of the May-Leonard asymmetric system (3)

$$\begin{aligned}\dot{x} &= x(1 - x - \alpha_1 y - \beta_1 z) = X(x, y, z),\\ \dot{y} &= y(1 - \beta_2 x - y - \alpha_2 z) = Y(x, y, z),\\ \dot{z} &= z(1 - \alpha_3 x - \beta_3 y - z) = Z(x, y, z),\end{aligned}$$

such that the system possess irreducible invariant surface of degree two (Darboux polynomial of degree two).

Invariant surfaces of the May-Leonard asymmetric system

Objective

Find conditions on the parameters of the May-Leonard asymmetric system (3)

$$\begin{aligned}\dot{x} =& x(1 - x - \alpha_1 y - \beta_1 z) = X(x, y, z),\\ \dot{y} =& y(1 - \beta_2 x - y - \alpha_2 z) = Y(x, y, z),\\ \dot{z} =& z(1 - \alpha_3 x - \beta_3 y - z) = Z(x, y, z),\end{aligned}$$

such that the system possess irreducible invariant surface of degree two (Darboux polynomial of degree two).

We look separately for invariant surfaces passing and not passing through the origin

Theorem

System (3) has an irreducible invariant surface of degree 2 not passing through the origin if one of the following 17 conditions or conjugated to it holds:

(1) $\alpha_2 = \beta_1 = \beta_2 - 1/2 = \alpha_1 - 3 = 0,$

(2) $\alpha_2 = \beta_1 = \beta_2 - 3 = \alpha_1 - 3 = 0$,

(3) $\beta_3 = \beta_1 = \alpha_3 + \beta_2 - 1 = \alpha_2 + 1 = \alpha_1 - \alpha_3 - 1 = 0,$

...

(15) $\beta_3 - 1/2 = \beta_2 - 3 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - 1/2 = 0,$

(16) $\beta_3 - 3 = \alpha_3 - 3 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0,$

(17) $\beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - \alpha_3 + 2 = 0.$

Proof of the theorem

A general form of a possible invariant surface of degree two of system (3) is

Proof of the theorem

A general form of a possible invariant surface of degree two of system (3) is

$$f(x,y,z) = h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2, \tag{9}$$

Proof of the theorem

A general form of a possible invariant surface of degree two of system (3) is

$$f(x,y,z) = h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2, \tag{9}$$

and its cofactor should be

Proof of the theorem

A general form of a possible invariant surface of degree two of system (3) is

$$f(x,y,z) = h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2, \tag{9}$$

and its cofactor should be

$$K(x,y,z) = c_0 + c_1 x + c_2 y + c_3 z. \tag{10}$$

Polynomial (9) will be an invariant surface of system (3) with cofactor (10) if

$$\mathfrak{X}f = \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial z}Z = Kf. \tag{11}$$

Polynomial (9) will be an invariant surface of system (3) with cofactor (10) if

$$\mathfrak{X}f = \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial z}Z = Kf. \tag{11}$$

Comparing the coefficients of similar terms in (11),

Polynomial (9) will be an invariant surface of system (3) with cofactor (10) if

$$\mathfrak{X}f = \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial z}Z = Kf. \qquad (11)$$

Comparing the coefficients of similar terms in (11), we obtain the polynomial conditions for existence of an invariant surface (9),

$$g_1 = g_2 = \ldots = g_{19} = g_{20} = 0,$$

where

$$
\begin{aligned}
g_1 &= -c_0 h_{000}, \\
g_2 &= -c_3 h_{000} + h_{001} - c_0 h_{001}, \\
&\ldots \\
g_{17} &= -h_{100} - c_1 h_{100} + 2h_{200} - c_0 h_{200}, \\
g_{18} &= -2h_{200} - c_1 h_{200}, \\
g_{19} &= -h_{110} - \beta_2 h_{110} - c_1 h_{110} - 2\alpha_1 h_{200} - c_2 h_{200}, \\
g_{20} &= -h_{101} - \alpha_3 h_{101} - c_1 h_{101} - 2\beta_1 h_{200} - c_3 h_{200}.
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
g_1 &= -c_0 h_{000}, \\
g_2 &= -c_3 h_{000} + h_{001} - c_0 h_{001}, \\
&\dots \\
g_{17} &= -h_{100} - c_1 h_{100} + 2h_{200} - c_0 h_{200}, \\
g_{18} &= -2h_{200} - c_1 h_{200}, \\
g_{19} &= -h_{110} - \beta_2 h_{110} - c_1 h_{110} - 2\alpha_1 h_{200} - c_2 h_{200}, \\
g_{20} &= -h_{101} - \alpha_3 h_{101} - c_1 h_{101} - 2\beta_1 h_{200} - c_3 h_{200}.
\end{aligned}
\tag{12}
$$

Let $I = \langle g_1, g_2, ..., g_{19}, g_{20} \rangle$ be the ideal generated by g_i.

- We have to find α_i, β_i such that the system $g_1 = g_2 = \cdots = g_{19} = g_{20} = 0$ has a solution.

$$
\begin{aligned}
g_1 &= -c_0 h_{000},\\
g_2 &= -c_3 h_{000} + h_{001} - c_0 h_{001},\\
&\dots\\
g_{17} &= -h_{100} - c_1 h_{100} + 2h_{200} - c_0 h_{200},\\
g_{18} &= -2h_{200} - c_1 h_{200},\\
g_{19} &= -h_{110} - \beta_2 h_{110} - c_1 h_{110} - 2\alpha_1 h_{200} - c_2 h_{200},\\
g_{20} &= -h_{101} - \alpha_3 h_{101} - c_1 h_{101} - 2\beta_1 h_{200} - c_3 h_{200}.
\end{aligned} \tag{12}
$$

Let $I = \langle g_1, g_2, ..., g_{19}, g_{20}\rangle$ be the ideal generated by g_i.

- We have to find α_i, β_i such that the system $g_1 = g_2 = \cdots = g_{19} = g_{20} = 0$ has a solution.
- To find such α_i, β_i it is sufficient to eliminate c_i, h_{jkm} from the ideal I.

To simplify the computations we consider separately the cases:

To simplify the computations we consider separately the cases:

- $h_{000} = 1$, meaning invariant curves $f = 0$ not passing through the origin;

To simplify the computations we consider separately the cases:

- $h_{000} = 1$, meaning invariant curves $f = 0$ not passing through the origin;
- $h_{000} = 0$, meaning invariant curves $f = 0$ passing through the origin.

To simplify the computations we consider separately the cases:

- $h_{000} = 1$, meaning invariant curves $f = 0$ not passing through the origin;
- $h_{000} = 0$, meaning invariant curves $f = 0$ passing through the origin.

From now on assume $h_{000} = 1$.

To find invariant surfaces of degree two, at least one coefficient

$$h_{200}, h_{110}, h_{101}, h_{020}, h_{011}, h_{002}$$

MUST BE different from zero.

It can be written in the six polynomial form as

- $1 - wh_{200} = 0;$
- $1 - wh_{110} = 0;$
- $1 - wh_{101} = 0;$
- $1 - wh_{020} = 0;$
- $1 - wh_{011} = 0;$
- $1 - wh_{002} = 0.$

with w being a new variable.

How to find systems admitting invariant surfaces with $h_{200} \neq 0$?

How to find systems admitting invariant surfaces with $h_{200} \neq 0$?

• Compute (e.g. using the routine `eliminate` of SINGULAR) the 13-th elimination ideal of the ideal

$$I^{(1)} = \langle I, 1 - wh_{200} \rangle,$$

in the ring

$$\mathbb{Q}[w, c_0, c_1, c_2, c_3, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3].$$

How to find systems admitting invariant surfaces with $h_{200} \neq 0$?

• Compute (e.g. using the routine `eliminate` of SINGULAR) the 13-th elimination ideal of the ideal

$$I^{(1)} = \langle I, 1 - wh_{200} \rangle,$$

in the ring

$$\mathbb{Q}[w, c_0, c_1, c_2, c_3, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3].$$

• Denote this elimination ideal by $I_{13}^{(1)}$; and its variety by $V_1 = \mathbf{V}(I_{13}^{(1)})$.

How to find all the possibles invariant surfaces?

How to find all the possibles invariant surfaces?

- Proceeding analogously, we can find other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$.

How to find all the possibles invariant surfaces?

• Proceeding analogously, we can find other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$.

Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, V_6 = \mathbf{V}(I_{13}^{(6)})$.

How to find all the possibles invariant surfaces?

• Proceeding analogously, we can find other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$.

Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, V_6 = \mathbf{V}(I_{13}^{(6)})$.

• The union, $V = V_1 \cup ... \cup V_6$, contains the set of all May-Leonard asymmetric systems, (3), having invariant surfaces of degree two not passing through the origin.

How to find all the possibles invariant surfaces?

- Proceeding analogously, we can find other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$.

Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, V_6 = \mathbf{V}(I_{13}^{(6)})$.

- The union, $V = V_1 \cup ... \cup V_6$, contains the set of all May-Leonard asymmetric systems, (3), having invariant surfaces of degree two not passing through the origin.

- Since $V = \mathbf{V}(I_{13}^{(1)}) \cup \cdots \cup \mathbf{V}(I_{13}^{(6)}) = \mathbf{V}(I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)})$, the irreducible decomposition of V is obtained:

How to find all the possibles invariant surfaces?

• Proceeding analogously, we can find other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$.

Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, V_6 = \mathbf{V}(I_{13}^{(6)})$.

• The union, $V = V_1 \cup ... \cup V_6$, contains the set of all May-Leonard asymmetric systems, (3), having invariant surfaces of degree two not passing through the origin.

• Since $V = \mathbf{V}(I_{13}^{(1)}) \cup \cdots \cup \mathbf{V}(I_{13}^{(6)}) = \mathbf{V}(I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)})$, the irreducible decomposition of V is obtained:

- computing the ideal $J = I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)}$ (routine `intersect` of SINGULAR);
- finding the irreducible decomposition of $\mathbf{V}(J)$ (routine `minAssGTZ` of SINGULAR).

The output gives us 88 ideals composing the irreducible decomposition of the variety $\mathbf{V}(J)$.

So there are 88 conditions on the parameters α_i, β_i of the May-Leonard asymmetric system for existence of an invariant surface of degree two not passing through the origin.

The May-Leonard asymmetric system has a symmetry with respect to change of axes

$$x \to z, \; y \to x, \; z \to y, \tag{13}$$

$$x \to y, \; y \to z, \; z \to x, \tag{14}$$

$$x \to y, \; y \to x, \; z \to z, \tag{15}$$

$$x \to z, \; y \to y, \; z \to z, \tag{16}$$

$$x \to x, \; y \to z, \; z \to y, \tag{17}$$

that do not change the shape of the system.

For instance, under transformation (13), $x \to z,\ y \to x,\ z \to y$, system (3) becomes

$$\begin{aligned}\dot{x} =& x(1 - x - \alpha_2 y - \beta_2 z),\\ \dot{y} =& y(1 - \beta_3 x - y - \alpha_3 z),\\ \dot{z} =& z(1 - \alpha_1 x - \beta_1 y - z),\end{aligned}$$

that can be obtained from system (3) by the change of parameters

$$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2. \tag{18}$$

Thus, if we have a condition on the parameters of (3) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (18).

Thus, if we have a condition on the parameters of (3) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (18).

For example, under condition (5) of Theorem 1,
$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0$,
system (3) has the invariant surface $f = 2 - 4x + 2x^2 - 2y + yz$.

Thus, if we have a condition on the parameters of (3) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (18).

For example, under condition (5) of Theorem 1,
$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0$,
system (3) has the invariant surface $f = 2 - 4x + 2x^2 - 2y + yz$.
Applying the transformation of parameters (18),
$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2$,
condition (5) becomes
$\beta_2 = \beta_3 = \alpha_2 + 1 = \beta_1 - 3 = \alpha_1 + 1 = \alpha_3 - 1/2 = 0$.

Thus, if we have a condition on the parameters of (3) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (18).

For example, under condition (5) of Theorem 1,
$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0$,
system (3) has the invariant surface $f = 2 - 4x + 2x^2 - 2y + yz$.
Applying the transformation of parameters (18),
$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2$,
condition (5) becomes
$\beta_2 = \beta_3 = \alpha_2 + 1 = \beta_1 - 3 = \alpha_1 + 1 = \alpha_3 - 1/2 = 0$.
This condition is one of the 88 obtained and gives us the invariant surface $f = 2 - 4z + 2z^2 - 2x + xy$ for system (3) under transformation (13).

Similarly, after transformations (13)–(17) the conditions for existence of invariant surfaces are changed according to the rules

Similarly, after transformations (13)–(17) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \quad (19)$$

$$\alpha_1 \to \beta_2,\ \beta_1 \to \ \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \quad (20)$$

$$\alpha_1 \to \beta_3,\ \beta_1 \to \ \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \quad (21)$$

$$\alpha_1 \to \beta_1,\ \beta_1 \to \ \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2. \quad (22)$$

Similarly, after transformations (13)–(17) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \quad (19)$$
$$\alpha_1 \to \beta_2,\ \beta_1 \to \ \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \quad (20)$$
$$\alpha_1 \to \beta_3,\ \beta_1 \to \ \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \quad (21)$$
$$\alpha_1 \to \beta_1,\ \beta_1 \to \ \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2. \quad (22)$$

We say, that two conditions for existence of invariant surfaces are **conjugate** if one can be obtained from another by means one of transformations (18)–(22).

Similarly, after transformations (13)–(17) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \quad (19)$$
$$\alpha_1 \to \beta_2,\ \beta_1 \to \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \quad (20)$$
$$\alpha_1 \to \beta_3,\ \beta_1 \to \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \quad (21)$$
$$\alpha_1 \to \beta_1,\ \beta_1 \to \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2. \quad (22)$$

We say, that two conditions for existence of invariant surfaces are **conjugate** if one can be obtained from another by means one of transformations (18)–(22).

We investigate the existence of invariant surfaces only for conditions which are not conjugated, reducing considerably the number of cases.

Moreover, some of that conditions provides invariant surfaces of degree two which are not irreducible, namely:

Moreover, some of that conditions provides invariant surfaces of degree two which are not irreducible, namely:

- $\alpha_1 = \beta_1 = 0$, providing the invariant surface $(1-x)^2$ (and its respective conjugated conditions);

Moreover, some of that conditions provides invariant surfaces of degree two which are not irreducible, namely:

- $\alpha_1 = \beta_1 = 0$, providing the invariant surface $(1 - x)^2$ (and its respective conjugated conditions);
- $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ providing the invariant surface $(1 - x - z)^2$ (and its respective conjugated conditions);

Moreover, some of that conditions provides invariant surfaces of degree two which are not irreducible, namely:

- $\alpha_1 = \beta_1 = 0$, providing the invariant surface $(1-x)^2$ (and its respective conjugated conditions);
- $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ providing the invariant surface $(1-x-z)^2$ (and its respective conjugated conditions);
- $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_2 - 2 = 0$ providing the invariant surface $(1-x-y-z)^2$ (and its respective conjugated conditions).

Moreover, some of that conditions provides invariant surfaces of degree two which are not irreducible, namely:

- $\alpha_1 = \beta_1 = 0$, providing the invariant surface $(1-x)^2$ (and its respective conjugated conditions);
- $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ providing the invariant surface $(1-x-z)^2$ (and its respective conjugated conditions);
- $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_2 - 2 = 0$ providing the invariant surface $(1-x-y-z)^2$ (and its respective conjugated conditions).

The remaining conditions are the 17 on the statement Theorem.

The corresponding surfaces are:

(1) $f = 1 - x - 2y + y^2$; (2) $f = 1 - 2x + x^2 - 2y - 2xy + y^2$;
(3) $f = 2 - 2x - 2y + yz$; (4) $f = 2 - 4x + 2x^2 - 2y + yz$;
(5) $f = 2 - 4x + 2x^2 + 2xy - 2z + xz$;
(6) $f = 2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz$;
(7) $f = 1 - x - 2y + y^2 + yz$; (8) $f = 2 - 4x + 2x^2 - 2y - 2z + xz$;
(9) $f = 1 - 2x + x^2 - 2y - 2xy + y^2 + yz$;
(10) $f = 1 - 2x + x^2 - y - z + xz$;
(11) $f = 1 - x - y - z + xz$; (12) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 + yz$;
(13) $f = 1 - x - y - 2z + z^2$; (14) $f = 1 - 2x + x^2 - y - 2z - 2xz + z^2$;
(15) $f = 1 - 2x + x^2 - y - 2z + 2xz + z^2$;
(16) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z - 2xz - 2yz + z^2$;
(17) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z + 2xz - 2yz + z^2$;

The corresponding surfaces are:

(1) $f = 1 - x - 2y + y^2$; (2) $f = 1 - 2x + x^2 - 2y - 2xy + y^2$;
(3) $f = 2 - 2x - 2y + yz$; (4) $f = 2 - 4x + 2x^2 - 2y + yz$;
(5) $f = 2 - 4x + 2x^2 + 2xy - 2z + xz$;
(6) $f = 2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz$;
(7) $f = 1 - x - 2y + y^2 + yz$; (8) $f = 2 - 4x + 2x^2 - 2y - 2z + xz$;
(9) $f = 1 - 2x + x^2 - 2y - 2xy + y^2 + yz$;
(10) $f = 1 - 2x + x^2 - y - z + xz$;
(11) $f = 1 - x - y - z + xz$; (12) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 + yz$;
(13) $f = 1 - x - y - 2z + z^2$; (14) $f = 1 - 2x + x^2 - y - 2z - 2xz + z^2$;
(15) $f = 1 - 2x + x^2 - y - 2z + 2xz + z^2$;
(16) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z - 2xz - 2yz + z^2$;
(17) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z + 2xz - 2yz + z^2$;

The proof is completed.

First integrals of the May-Leonard asymmetric system

Objective

Construct Darboux first integrals for the May-Leonard asymmetric system, (3), for the 17 families of Theorem 1 using invariant planes and invariant surfaces of degree two.

Theorem

a) If one of conditions 1-3, 11, 12, 17 of Theorem 1 holds, then the corresponding system (3) *admits at least one Darboux first integral.*
b) If one of conditions 4-10, 13-16 of Theorem 1 holds, then the corresponding system (3) *is completely integrable, that is, it admits two independents Darboux first integrals.*

a) Condition (1):

a) Condition (1): the May-Leonard system has the form

$$\dot{x} = x(1 - x - 3y), \quad \dot{y} = y(1 - x/2 - y)y, \quad \dot{z} = z(1 - \alpha_3 x - \beta_3 y - z). \tag{23}$$

Using invariant planes and invariant surfaces of degree two, by the Darboux theory we have the Darboux first integral

$$\widetilde{H} = x^{\lambda_1} y^{\lambda_2} (x + 4y)^{\lambda_1} \left(x - 2y^2 + 2y\right)^{-\lambda_2 - 2\lambda_1} \left(-x + y^2 - 2y + 1\right)^{\frac{\lambda_2}{2}},$$

λ_1, λ_2 not both equal to zero.

a) Condition (1): the May-Leonard system has the form

$$\dot{x} = x(1-x-3y), \quad \dot{y} = y(1-x/2-y)y, \quad \dot{z} = z(1-\alpha_3 x-\beta_3 y-z). \tag{23}$$

Using invariant planes and invariant surfaces of degree two, by the Darboux theory we have the Darboux first integral

$$\widetilde{H} = x^{\lambda_1} y^{\lambda_2} (x+4y)^{\lambda_1} \left(x - 2y^2 + 2y\right)^{-\lambda_2-2\lambda_1} \left(-x + y^2 - 2y + 1\right)^{\frac{\lambda_2}{2}},$$

λ_1, λ_2 not both equal to zero.

In particular, taking $\lambda_4 = 1$ and $= \lambda_3 = 0$ we have the Darboux first integral

$$H = \frac{x(x+4y)}{(x+2y-2y^2)^2}.$$

Since first two equations of (23) are independent of z we cannot construct another independent first integral $H_2(x, y)$ of (23) using only such surfaces.

Since first two equations of (23) are independent of z we cannot construct another independent first integral $H_2(x,y)$ of (23) using only such surfaces. If such integral would exist then the two-dimensional system

$$\dot{x} = x(1 - x - 3y), \quad \dot{y} = y(1 - x/2 - y)y,$$

would have two independent first integrals, which is impossible.

b) Condition (4): Using invariant planes and invariant surfaces of degree two, by the Darboux theory we have the Darboux first integrals

$$H_1 = \frac{z(2 - 4x + 2x^2 - 2y + yz)}{(4x + y - 2z)},$$

$$H_2 = \frac{yz}{x^2}.$$

To check if these first integrals are independent we compute their gradients

To check if these first integrals are independent we compute their gradients

$$G_1 = \{\frac{4(-2+2x+y)(1+x-z)z}{(4x+y-2z)^2},\ -\frac{2(1+x-z)^2 z}{(4x+y-2z)^2},\ \frac{2(4x-8x^2+4x^3+y-6xy+x^2y-y^2+4xyz+y^2z-yz^2)}{(4x+y-2z)^2}\},$$

$$G_2 = \{-\frac{2yz}{x^3},\ \frac{z}{x^2},\ \frac{y}{x^2}\},$$

To check if these first integrals are independent we compute their gradients

$$G_1 = \{\frac{4(-2+2x+y)(1+x-z)z}{(4x+y-2z)^2},\ -\frac{2(1+x-z)^2 z}{(4x+y-2z)^2},$$
$$\frac{2(4x-8x^2+4x^3+y-6xy+x^2y-y^2+4xyz+y^2z-yz^2)}{(4x+y-2z)^2}\},$$

$$G_2 = \{-\frac{2yz}{x^3},\ \frac{z}{x^2},\ \frac{y}{x^2}\},$$

and we verified that the linear combination $aG_1 + bG_2$, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

To check if these first integrals are independent we compute their gradients

$$G_1 = \{\frac{4(-2+2x+y)(1+x-z)z}{(4x+y-2z)^2},\ -\frac{2(1+x-z)^2 z}{(4x+y-2z)^2},\ \frac{2(4x-8x^2+4x^3+y-6xy+x^2y-y^2+4xyz+y^2z-yz^2)}{(4x+y-2z)^2}\},$$

$$G_2 = \{-\frac{2yz}{x^3},\ \frac{z}{x^2},\ \frac{y}{x^2}\},$$

and we verified that the linear combination $aG_1 + bG_2$, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

Therefore the Darboux first integrals H_1 and H_2 are independents.

Conclusions

Using invariant surfaces of degree two and invariant planes:

Conclusions

Using invariant surfaces of degree two and invariant planes:

- 6 families of the May-Leonard asymmetric system admitting at least one Darboux first integral are found;

Conclusions

Using invariant surfaces of degree two and invariant planes:

- 6 families of the May-Leonard asymmetric system admitting at least one Darboux first integral are found;
- 11 families of the May-Leonard asymmetric system admitting two independents Darboux first integral are found;

Conclusions

Using invariant surfaces of degree two and invariant planes:

- 6 families of the May-Leonard asymmetric system admitting at least one Darboux first integral are found;
- 11 families of the May-Leonard asymmetric system admitting two independents Darboux first integral are found;
- we could not find invariant surfaces passing through the origin, that is, polynomials (9) with $h_{000} = 0$;

Conclusions

Using invariant surfaces of degree two and invariant planes:

- 6 families of the May-Leonard asymmetric system admitting at least one Darboux first integral are found;
- 11 families of the May-Leonard asymmetric system admitting two independents Darboux first integral are found;
- we could not find invariant surfaces passing through the origin, that is, polynomials (9) with $h_{000} = 0$;
  These case is computationally more difficult since the system has 3 invariant planes $x = 0$, $y = 0$, $z = 0$ passing through the origin and this yields a complicate structure of the corresponding ideal I.

The work was supported by
the Slovenian Research Agency and
by FP7-PEOPLE-2012-IRSES-316338

The work was supported by
the Slovenian Research Agency and
by FP7-PEOPLE-2012-IRSES-316338

Thank you for your attention!