# CONTROLLABILITY AND PUGH'S CLOSING LEMMA FOR SOME FLOWS WITH INFINITE INVARIANT MEASURES


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# Basic definitions from topological dynamics

Consider an autonomous ODE system

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d, \quad d \geq 3. \tag{1}$$

where $V$ is uniformly continuous, bounded and uniquely integrable i.e. any Cauchy problem has a unique solution. For instance, we may take $V$ uniformly Lipschitz continuous.

Let $\varphi_V(t, x_0) = x(t, 0, x_0)$ be the flow of (1). Let $|\cdot|$ be the Euclidean norm.

Mainly we assume that

$$\|V\|_{C^1} < \infty$$

The majority of facts and proofs we list later on are also true for $d = 1, 2$.

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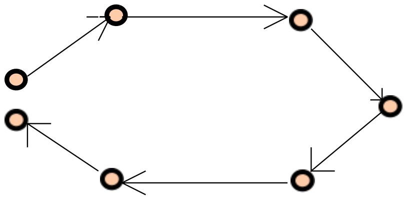
## Poisson stable points $\mathrm{R}(V)$

A point $x_0 \in \mathbb{R}^d$ is called Poisson stable or, sometimes recurrent, if for any $\varepsilon > 0$ there is $t_0 > 0$ such that $|\varphi_V(t_0, x_0) - x_0| < \varepsilon$. The point $x$ is $\omega$ - limit for itself.

## Remark

It is sufficient for us to consider positive direction stability only.



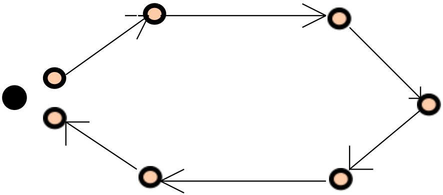
# Basic definitions from topological dynamics

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d, \quad d \geq 3. \tag{1}$$

Let $\varphi_V(t, x_0) = x(t, 0, x_0)$ be the flow of (1). Let $|\cdot|$ be the Euclidean norm.

## Non-wandering points $\Omega(V)$

A point $x_0 \in \mathbb{R}^d$ is called non-wandering if for any neighbourhood $U \ni x_0$ and any $t_0 > 0$ there exist $t > t_0$ and two points $p, q \in U$ such that $q = \varphi_T(t, p)$.



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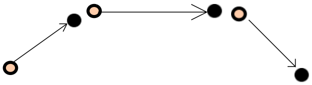
## Pseudotrajectories

We consider $\delta$ - solutions of system (1) that $C^1$ smooth functions $x(t)$ such that $|\dot{x}(t) - V(x(t))| < \delta$.



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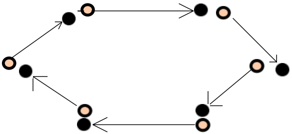
Let $\varphi_V(t, x_0) = x(t, 0, x_0)$ be the flow of (1). Let $|\cdot|$ be the Euclidean norm.

## Chain recurrence $CR(V)$

A point $x_0 \in \mathbb{R}^d$ is called chain recurrent if for any $\delta > 0$ there is a periodic $\delta$ – solution $x(t)$ with $x(0) = x_0$.



# Basic definitions from topological dynamics

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## Inclusions

Sets $\Omega(V)$ and $\mathrm{CR}(V)$ are closed and

$$\overline{\mathrm{R}(V)} \subset \Omega(V) \subset \mathrm{CR}(V)$$

Here overline stands for closure of a set.

A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1997.

# Basic definitions from topological dynamics

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d, \quad d \geq 3. \tag{1}$$

Let $\varphi_V(t, x_0) = x(t, 0, x_0)$ be the flow of (1). Let $|\cdot|$ be the Euclidean norm.

## Chain transitivity

**System (1)** is called controllable if for any $\delta > 0$ and any two points $p$ and $q$ there is a $\delta$ – solution $x(t)$ and $T > 0$ with $x(0) = p$, $x(T) = q$.

Controllability $\Rightarrow$ no sinks, sources and attractors/repellers in general.
Another counterexample for controllability is given by o.d.e. $\dot{x} = 1$ where the sink (and, also the source) is infinity.

# Basic definitions from topological dynamics

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d, \quad d \geq 3. \tag{1}$$

Let $\varphi_V(t, x_0) = x(t, 0, x_0)$ be the flow of (1). Let $|\cdot|$ be the Euclidean norm.

## Invariant measures

A Borel locally finite measure $\mu$ is called invariant with respect to Eq. (1) if $\mu(\varphi_V(t, A)) = \mu(A)$ for $t \in \mathbb{R}$ and any measurable set $A$.

If the vector field $V$ is smooth, the measure $\mu$ is absolutely continuous w.r.t. to standard Lebesgue measure $d\mu(x) = p(x)\, dm_d(x)$ with the smooth density $p(x)$ this measure is invariant iff

$$\mathrm{div}(p(x)V(x)) = 0 \tag{2}$$

for any $x \in \mathbb{R}^d$.

# Basic definitions from topological dynamics

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## Invariant measures

A Borel locally finite measure $\mu$ is called invariant with respect to Eq. (1) if $\mu(\varphi_V(t, A)) = \mu(A)$ for $t \in \mathbb{R}$ and any measurable set $A$.

## Support of the measure

$$\operatorname{supp}(\mu) = \bigcap \{A : A \text{ is closed}, \quad \mu(\mathbb{R}^d \setminus A) = 0\}.$$

$x \in \operatorname{supp}(\mu)$ iff $\mu(B(x, \varepsilon) > 0)$ for any $\varepsilon > 0$.

# Some classical results

## Poincaré recurrence theorem

Let system (1) preserves a probability measure $\mu$ (i.e. this measure is invariant). Then for any measurable set $A$ and any $T > 0$ we have

$$\mu(\{x_0 \in A : \{\varphi_V(t, x_0)\}_{t \geq T}\} \subset X \setminus A) = 0.$$

# Some classical results

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$$\mu(\{x_0 \in A : \{\varphi_V(t, x_0)\}_{t \geq T}\} \subset \mathbb{R}^d \setminus A) = 0.$$

For non-probability measures it is in general wrong e.g. $\dot{x} = 1$ on $\mathbb{R}$.

# Some classical results

## Poincaré recurrence theorem

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$$\mu(\{x_0 \in A : \{\varphi_V(t, x_0)\}_{t \geq T}\} \subset \mathbb{R}^d \setminus A) = 0.$$

## Corollary

Let $\mu$ be a Borel probability invariant measure. Then

$$\mathrm{supp}\,(\mu) \subset \overline{\mathrm{R}(V)}.$$

# Some classical results

## Chain recurrence and chain transitivity.

### Proposition

Let $V$ be continuous, bounded, uniquely integrable and all points of $\mathbb{R}^d$ such that all its points are chain recurrent. Then the flow $\varphi_V$ is chain transitive.

Similar to

### Proposition

Let $T$ be a homeorphism of a connected metric space $X$ such that all its points are chain recurrent. Then the space $X$ is chain transitive.

S. Crovisier, Perturbation of $C^1$-diffeomorphisms and generic conservative dynamics on surfaces, 2006.

# Closing and connecting lemmas

## Pugh's closing lemma

Let $V$ be a $C^1$ - bounded smooth vector field with the uniformly bounded Jacobi matrix. Let $x \in \Omega(V)$. Then for any $\varepsilon > 0$ there exists a $C^1$ smooth vector field $W : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\|W\|_{C^1} < \varepsilon \tag{3}$$

and $x$ is a periodic point for the system

$$\dot{x} = V(x) + W(x). \tag{4}$$

C. Pugh, The closing lemma. Amer. J. Math. **89** (1967), 956–1009.

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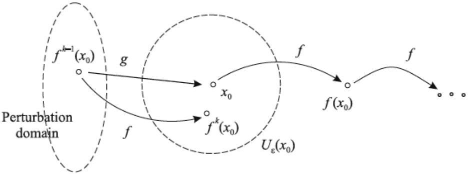


The local perturbation of the vector fields in a point should not touch a given segment of trajectory of the point.

# Closing and connecting lemmas

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C. Pugh, The closing lemma. Amer. J. Math. **89** (1967), 956–1009.

No direct $C^2$ analog:
C. Gutierrez, A counter-example to a $C^2$ closing lemma. Ergodic Theory Dynam. Systems 7 (1987), no. 4, 509–530.

# Closing and connecting lemmas

## Definition.

Let $T$ be a diffeomorphism of a compact Riemannian manifold with distance $\rho$. We say that a point $x \in M$ is well-closable if for any $\varepsilon > 0$ there exists a sequence of diffeomorphisms $S_n$, converging to $T$ in $\mathrm{Diff}^1(M)$ and such that $x$ is a $t_n$ – periodic point for every map $S_n$ such that $\rho(T^k(x), S_n^k(x)) < \varepsilon$, $k = 1, \dots, t_n$.

## Mañé ergodic closing lemma.

For any diffeomorphism $T$ of a Riemannian manifold $M$ and for every Borel probability $T$–invariant measure the set of all well-closable points has full invariant measure.

R. Mañé, An ergodic closing lemma, Ann. of Math. (2) **116** (1982), 503–540.

# Closing and connecting lemmas

For Hamiltonian systems the closing perturbation could also be made Hamiltonian.


C. Pugh, C. Robinson, The $C^1$ closing lemma, including Hamiltonians.Ergodic Theory Dynam. Systems 3 (1983), no. 2, 261–313.

M. Herman, Différentiabilité optimale et contre-exemples à la fermeture en topologie $C^\infty$ des orbites récurrentes de flots hamiltoniens. C. R. Acad. Sci. Paris Sér. I Math. 313 (1991), no. 1, 49–51.

M. Asaoka, K. Irie, A $C^\infty$ closing lemma for Hamiltonian diffeomorphisms of closed surfaces. Geometric and Functional Analysis. 26 (2016) 1245–1254. doi:10.1007/s00039-016-0386-3.

# Closing and connecting lemmas

## Pugh's closing lemma

Let $V$ be a $C^1$ - bounded smooth vector field with the uniformly bounded Jacobi matrix. Let $x \in \Omega(V)$. Then for any $\varepsilon > 0$ there exists a $C^1$ smooth vector field $W : \mathbb{R}^d \to \mathbb{R}^d$ such that

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The minimal requirement on the point $x$ is that it is chain recurrent. Is this condition sufficient?

# Closing and connecting lemmas

## Pugh's closing lemma

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The minimal requirement on the point $x$ is that it is chain recurrent. Is this condition sufficient?

Almost yes and even a bit better!

# Closing and connecting lemmas

M.-C. Arnaud, Ch. Bonatti, S. Crovisier, Dynamiques symplectiques génériques, Ergodic Theory Dynam. Systems 25 (2005), no. 5, 1401–1436.

Ch. Bonatti, S. Crovisier, Récurrence et généricité, Invent. Math. 158 (2004), no. 1, 33–104.

Sh. Hayashi, Connecting invariant manifolds and the solution of the $C^1$ stability and Ω–stability conjectures for flows, Ann. of Math. (2) 145 (1997), no. 1, 81–137.

Lan Wen and Zhihong Xia, $C^1$ - Connecting Lemmas, Trans. Amer. Math. Soc., 2000, vol. 352, no. 11, pp. 5213?5230.

# Closing and connecting lemmas

## Pugh's closing lemma (discrete)

Let $T$ be a $C^1$ - diffeomorphism of a $C^1$ - smooth manifold $M$. Let $x \in \Omega(M, T)$. Then for any $\varepsilon > 0$ there exists a diffeomorphism $S : M \to M$ such that $\mathrm{dist}_{C^1}(S, T) < \varepsilon$ and $x$ is a periodic point for $S$.

## Connecting lemma by Bonatti and Crovisier.

Let $T$ be a $C^1$ – diffeomorphism of a smooth manifold $M$ whose periodic orbits are all hyperbolic. Consider points $x, y$ such that for every $\delta > 0$ there are $\delta$ – chains (non-singletons) starting at $x$ and ending at $y$. Then there is arbitrarily small $C^1$ perturbation $S$ of $T$ such that $y$ belongs to the positive $S$ – semiorbit of $x$.

In fact it suffices to suppose that eigenvalues of matrices $DT^n(x)$, $n$ is the period of $x$, do not equal roots of 1.

# Closing and connecting lemmas

Farther survey:


- D. V. Anosov, E. V. Zhuzhoma, Closing lemmas, Differential Equations **48:13**, (2012), 1653–1699.
- M.-C. Arnaud, Création de connexions en topologie $C^1$ pour les flots des surfaces, Bol. Soc. Mat., **30:3** (1999), 315–366.
- S. Crovisier, Perturbation of $C^1$ Diffeomorphisms and Generic Conservative Dynamics on Surfaces, Panor. Synth'eses, 2006, vol. 21, pp. 1–33.
- V. A. Pliss, A Certain Version of the Closing Lemma, Differ. Uravn., **7:5** (1971), 840–850.
- Jiehua Mai, A Simpler Proof of C1 Closing Lemma, Scienta Sinica (Ser. A), **29** (2006), 1021–1031.
- M. L. A. Peixoto, C. Pugh, The Planar Closing Lemma for Chain Recurrence, Trans. Amer. Math. Soc., **341**(1994), 173–192.
- C. Pugh, Special Cr Closing Lemma, Lecture Notes in Math., **1007**, (1999) 636–650.
- Lan Wen, A Uniform $C^1$ Connecting Lemma, Discrete Contin. Dyn. Syst., **8** (2002) 257–265.

# Poincaré's Theorem + Pugh's Lemma = ?.

## Theorem

Let the system (1) $\dot{x} = V(x)$ with the continuous bounded in $C^1$ vector field $V$ have an invariant measure $\mu$ such that $\operatorname{supp} \mu = \mathbb{R}^d$. Then for any $x \in \mathbb{R}^d$ and any $\varepsilon > 0$ there exists a $C^1$ smooth vector field $W : \mathbb{R}^d \to \mathbb{R}^d$ such that

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# Feeble fish

D. Burago, S. Ivanov, A. Novikov, A survival guide for feeble fish, arXiv:1605.01702v2.

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d. \tag{1}$$

Divergence-free

$$\operatorname{div} V = 0. \tag{5}$$

Boundedness:

$$\sup_x |V(x)| \leq M < +\infty. \tag{6}$$

Uniform Lipschitz continuity

$$|V(x) - V(y)| \leq K|x - y|. \tag{7}$$

Small mean drift condition

$$\lim_{L \to \infty} \sup_{x \in \mathbb{R}^d} \left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\, dy \right\| = 0. \tag{8}$$

# Feeble fish

D. Burago, S. Ivanov, A. Novikov, A survival guide for feeble fish, arXiv:1605.01702v2.

In fact, it was proved that if conditions (5)-(8) are fulfilled, system (1) is controllble.

## Perturbations.

For a given $\delta > 0$ we consider a piecewise continuous function $u : \mathbb{R} \to \mathbb{R}^d$ and the perturbed system

$$\dot{x} = V(x) + u(t). \tag{9}$$

Let $x_u(t, x_0)$ be the solution of Eq. (9) with initial conditions $x_u(0) = x_0$. Given a control $u(t)$ and two points $x_0$ and $y_0$ we take all values of $\theta > 0$ such that

$$x_u(\theta, x_0) = y_0$$

(the set of such $\theta$ may be empty). Given two points $x_0$ and $y_0$ we define $\tau_\delta(x_0, y_0)$ as the infimum of such values $\theta$ over the set of all controls $u(t) : |u(t)| \leq \delta$ (we set infimum of the empty set equal to infinity).

# Feeble fish

D. Burago, S. Ivanov, A. Novikov, A survival guide for feeble fish, arXiv:1605.01702v2.

## Theorem

Suppose the vector field $V$ of Eq.(1) satisfy conditions (5)–(8). Then $\tau(x,y) < \infty$ for all $x, y \in \mathbb{R}^d$ (we take $\delta = 1$). The following travel-time estimate

$$\tau_1(x,y) \leq C_1|x-y| + C_2, \quad (10)$$

holds with some $C_1$ and $C_2$ that depend on $V$ only.

To get the controllability ($\tau_\delta(x_0, y_0) < +\infty$) is suffices to assume that $V$ is locally Lipschitz continuous only.

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D. Burago, S. Ivanov, A. Novikov, A survival guide for feeble fish, arXiv:1605.01702v2.

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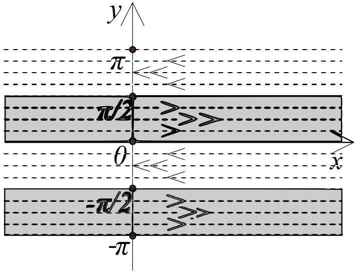
**Example:** $\dot{x} = \sin y$; $\dot{y} = 0$



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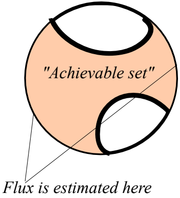
Idea of the proof: estimate the flux of the vector field on the border of "acheivable" set.



# Statement of the problem

## Main question

In conditions of the Burago - Ivanov - Novikov Theorem, could we say that closing/connecting lemmas are applicable?

# Main result

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d. \tag{1}$$

$$\operatorname{div} V = 0. \tag{5}$$

$$\lim_{L\to\infty} \sup_{x\in\mathbb{R}^d} \left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\, dy \right\| = 0. \tag{8}$$

$$\|V\|_{C^1} < +\infty. \tag{11}$$

## Theorem 1.

Let conditions (5), (8) and (11) be satisfied for Eq. (1). Then for any point $p \in \mathbb{R}^d$ and any $\varepsilon > 0$ there exists a vector field $W$: $\|W\|_{C^1} < \varepsilon$ such that the point $p$ becomes periodic with respect to system

$$\dot{x} = V(x) + W(x). \tag{4}$$

# Scheme of the proof

Recall the notion $B(x, r)$ – an $r$ - ball, centered at $x$.

## Modified Poincaré Theorem

Let the vector field $V$ be continuous and all solutions of system (1) be unique. Let $\mu$ be an invariant measure $\mu$ (that may be infinite) such that $\operatorname{supp}\mu = \mathbb{R}^d$. Suppose that there exists a continuous function $K : [0, \infty) \to [0, \infty)$ such that:

$$K(r) \geq \sup_{|x| \leq r} \frac{\langle V(x), x \rangle}{|x|}, \qquad r > 0;$$

$$\int_0^\infty \frac{1}{K(\rho)}\, d\rho = +\infty$$

and

$$\mu(B(0, r)) = o\left(\int_0^r \frac{1}{K(\rho)}\, d\rho\right), \qquad r \to \infty.$$

Then for any measurable set $A$ and any $N \in \mathbb{N}$ we have

$$\mu(\{x \in A : \{\varphi_V(t, x) : t > T\} \subset \mathbb{R}^d \setminus A\}) = 0.$$

# Scheme of the proof

Recall the notion $B(x, r)$ – an $r$ - ball, centered at $x$.

## Corollary

Let $\mu$ be a local finite Borel measure in $\mathbb{R}^d$, invariant with respect to Eq. (1), satisfying conditions (11) i.e. the vector field is bounded in $C^1$. Let

$$\mu(B(0, R)) = o(R)$$

as $R \to \infty$. Then $\Omega(V) = \mathbb{R}^d$.

# Scheme of the proof

## Selection of invariant measure for the perturbed system.

Later on we take $\mu$ as the measure with the density

$$p_A(x) := (x^2 + A)^{-\sigma}.$$

Here $A > 0$ is a big parameter, $\sigma \in ((d-1)/2, d/2)$. Then

$$\mu(B(0,R)) = 1 + \int_1^R r^{-2\sigma}\, dr = o(R).$$

The statement of the Modified Poincaré Theorem is fulfilled.

# Scheme of the proof

$p_A(x) := (x^2 + A)^{-\sigma}, \qquad A > 0$ is big, $\sigma \in ((d-1)/2, d/2)$.

## Constructing the perturbation-1

In order to have the measure $\mu$ with the given density $p_A(x)$ invariant w.r.t the perturbed system we must have

$$\operatorname{div}(p_A(x)(V(x) + W_A(x))) = 0 \text{ or } \operatorname{div}(p_A(x)W_A(x)) = -\operatorname{div}(p_A(x)V(x)).$$

# Scheme of the proof

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Or, since $\mathrm{div}\,V(x) = 0$, we have

$$\mathrm{div}\,(p_A(x)W_A(x)) = -\langle V(x), \nabla p_A(x)\rangle.$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product.

# Scheme of the proof

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Here $\langle\cdot,\cdot\rangle$ is the scalar product.

Suppose that $p_A(x)W(x) = \nabla U_A(x)$. Then $U_A(x)$ is the bounded solution of the Poisson equation

$$\Delta U = -\langle V(x), \nabla p_A(x)\rangle. \tag{12}$$

# Scheme of the proof

$p_A(x) := (x^2 + A)^{-\sigma}$ ,   $A > 0$ is big, $\sigma \in ((d-1)/2, d/2)$.

## Constructing the perturbation-2

Suppose that $p_A(x)W_A(x) = \nabla U_A(x)$. Then $U_A(x)$ is the bounded solution of the Poisson equation

$$\Delta U = -\langle V(x), \nabla p_A(x)\rangle. \qquad (12)$$

Then

$$U(x) = \int_{\mathbb{R}^d} \frac{C}{|x-y|^{d-2}} \langle V(x), \nabla p_A(x)\rangle \, dy$$

where $C$ is the constant that depends on $d$ only.

# Scheme of the proof

$p_A(x) := (x^2 + A)^{-\sigma}$ , $A > 0$ is big, $\sigma \in ((d-1)/2, d/2)$.

## Constructing the perturbation-2

Suppose that $p_A(x)W_A(x) = \nabla U_A(x)$. Then $U_A(x)$ is the bounded solution of the Poisson equation

$$\Delta U = -\langle V(x), \nabla p_A(x)\rangle. \tag{12}$$

Then

$$U_A(x) = \int_{\mathbb{R}^d} \frac{C}{|x-y|^{d-2}} \langle V(y), \nabla p_A(y)\rangle \, dy.$$

$$W_A(x) = \frac{C\nabla U_A(x)}{p_A(x)} = \frac{C}{p_A(x)} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \langle V(y), \nabla p_A(y)\rangle \, dy$$

## The most important part

$W_A(x)$ is $C^1$ - small if $A$ is big.

# Scheme of the proof

## Endgame

To finish the proof we notice that since the Poincaré recurrence theorem is applicable, all points are Poisson stable and thus the local Pugh's construction could be applicable.

# Comments and remarks

$$\dot{x} = V(x), \qquad x \in \mathbb{R}^d. \tag{1}$$

$$\operatorname{div} V = 0. \tag{5}$$

$$\lim_{L\to\infty} \sup_{x\in\mathbb{R}^d} \left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\, dy \right\| = 0. \tag{8}$$

$$\|V\|_{C^0} < +\infty. \tag{13}$$

## Theorem 2.

Let conditions (5), (8) and (13) be satisfied for Eq. (1) and $V$ be uniformly continuous. Then for any point $p \in \mathbb{R}^d$ and any $\varepsilon > 0$ there exists a vector field $W$: $\|W\|_{C^0} < \varepsilon$ such that the point $p$ becomes periodic with respect to system

$$\dot{x} = V(x) + W(x). \tag{4}$$

It suffices to replace the vector field $V$ with the convolution $V * h$ with am appropriate mollifier $h$.

# Comments and remarks

## Connecting conjecture.

Let all equilibria of Eq. (1) be hyperbolic. Then for all points $p, q \in \mathbb{R}^d$ and all $\varepsilon > 0$ there exists a perturbation $W$: $\|W\|_{C^1} < \varepsilon$ and $t > 0$ such that $q = \varphi_{V+W}(t, p)$.

Ch. Bonatti, S. Crovisier, Récurrence et généricité, Invent. Math. 158 (2004), no. 1, 33–104.

Techniques of this paper work in the non-compact case.

# Comments and remarks

## Advantages of our approach

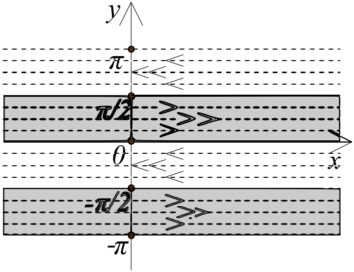
- Construction of an "almost probability" invariant measure with the Poincaré Recurrence Theorem applicable.
- Chain recurrent case with no requirements on hyperbolicity of periodic solutions.
- The proof does not use geometry of Euclidean spaces (we could work with general measures, distributed everywhere).

## Disadvantages

- Global perturbation.
- The perturbed system is not conservative anymore.
- Proof of the Connecting Lemma still requires hyperbolicity of all periodic points.

# Comments and remarks

**Example:** $\dot{x} = \sin y;\ \dot{y} = 0$




Observe that almost all points are wandering and all equilibria are non-hyperbolic. Neither they are not well-closable.

# Thank you!