Global attractors and their structure for scalar delay difference equations

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joint work with Christian Pötzsche

Delay Differential Equations (DDE)

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

$$\dot{x}(t) = f(x(t), x(t-\tau(x(t))))$$

$$\dot{x}(t) = f(t, x(t), x(t-\tau))$$

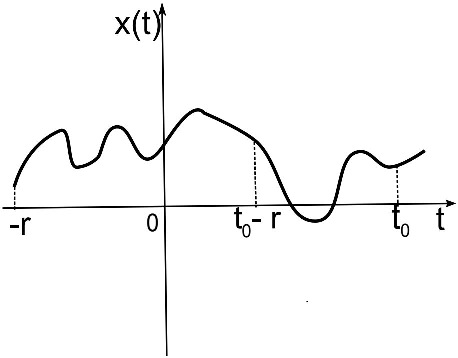
$\dot{x}(t) = f(t, x_t)$, where

$$x_t = x_{t,\tau}(\theta) = x(t+\theta) \quad \text{for all} \quad \theta \in [-\tau, 0]$$

$$\dot{x}(t) = h(x(t)) + \int_{t-\tau}^{t} g(x(t-s))\, ds$$

Examples of delay differential equations

$$\text{IVP}: \begin{cases} \dot{x}(t) = f(t, x(t), x(t-r)) \\ x_{0,r} = \varphi \end{cases}$$



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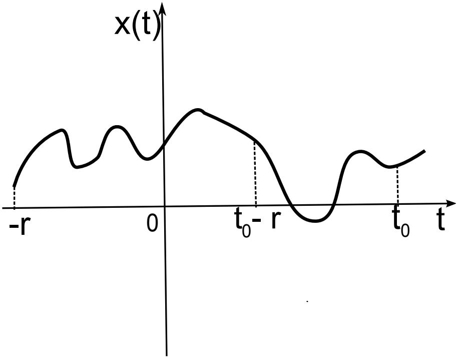
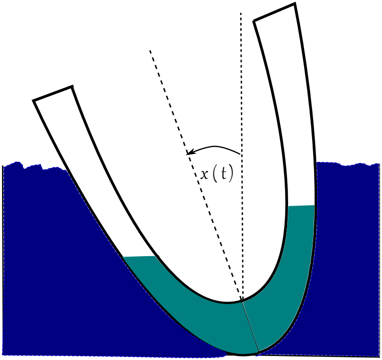


Definition

Phase space:
$C_r := C([-r,0],\mathbb{R})$
Let $x_{t,r} \in C_r$ be defined as
$x_{t,r}(\theta) = x(t+\theta)$
for all $\theta \in [-r,0]$.

Stabilizing warships...

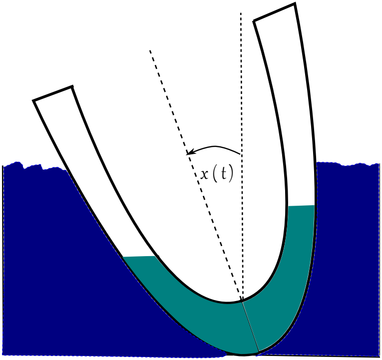


Simplified equation:

$$\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0;\ \ b, k > 0.$$

$x(t) = 0$ is a stable equilibrium point with convergence velocity $e^{-b/2}$.

Stabilizing warships...



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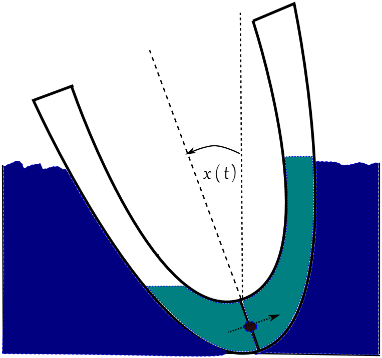
Idea: "Let us increase b using a servo-mechanism."

...without success.



$$\ddot{x}(t)+(b+q)\dot{x}(t)+kx(t) = 0;\ b, q, k > 0.$$

Due to the delay in the servo-mechanism:

$$\ddot{x}(t) + b\dot{x}(t) + q\dot{x}(t-\tau) + kx(t) = 0.$$

Taking the delay into account may induce:

Periodic oscillations,

Chaotic oscillations,

Syncronization, desyncronization,

Transient oscillations,

Change of the domain of attraction.

We consider

$$x_{k+1} = f(x_k, x_{k-d}), \tag{1}$$

or equivalently

$$y_{k+1} = F(y_k), \qquad F(y(0), \dots, y(d)) := \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ f(y(d), y(0)) \end{pmatrix} \tag{2}$$

where

- $d \geq 1$ integer (delay),
- $f \colon J^2 \to J$ with J a closed interval, f is C^1, $0 \in J$ inner point, $f(0,0) = 0$,
- $D_1 f(\xi, \eta) > 0$ for all $(\xi, \eta) \in J^2$,
- $f(0, \eta)\eta > 0$ (or $f(0, \eta)\eta < 0$) for all $\eta \neq 0$,
- $D_2 f(0,0) > 0$ (resp. $D_2 f(0,0) < 0$),
- the global attractor exists.

The state space for (2) is J^{d+1}.

Definition

A nonempty subset $\mathcal{A}$ of J^{d+1} is called a *global attractor* of (2) if:

(1) $\mathcal{A}$ is a closed, bounded subset of J^{d+1},

(2) $\mathcal{A}$ is invariant under F (i.e. $F(\mathcal{A}) = \mathcal{A}$),

(3) $\mathcal{A}$ attracts every bounded subset $B \subseteq J^{d+1}$ under iteration of F.

Provided the global attractor exists, it is unique and allows the dynamical characterization

$$\mathcal{A} = \left\{\eta \in J^{d+1} : \text{there is a bounded entire solution of (2) through } \eta\right\}.$$

Definition

A *Morse decomposition* of the global attractor $\mathcal{A}$ is a **finite, ordered collection**

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_m$$

of **disjoint**, **compact** and **invariant** *Morse sets* $\mathcal{M}_0, \ldots, \mathcal{M}_m \subseteq \mathcal{A}$ such that for every $\eta \in \mathcal{A}$ and for all bounded entire solution $y\colon \mathbb{Z} \to J^{d+1}$ for which $y_0 = \eta$ holds, there exist $i \geq j$ with

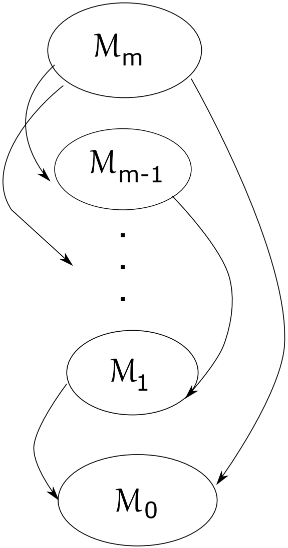
- $\alpha(y) \subseteq \mathcal{M}_i$ and $\omega(\eta) \subseteq \mathcal{M}_j$,
- $i = j \Rightarrow \eta \in \mathcal{M}_i$ (thus, $y_k \in \mathcal{M}_i$ for every $k \in \mathbb{Z}$).

The *connecting sets*

$$\mathcal{C}_j^i := \{\eta \in \mathcal{A} :\ \alpha(y) \subseteq \mathcal{M}_i \text{ and } \omega(\eta) \subseteq \mathcal{M}_j\} \text{ for all } j < i$$

consist of connecting orbits. Together with the Morse sets $\mathcal{M}_0, \ldots, \mathcal{M}_m$ they form a partition of $\mathcal{A}$.

Morse-decomposition



- Euler-discretization of DDEs $\dot{x}(t) = g(x(t), x(t-1))$

$$x_{k+1} = f(x_k, x_{k-d}) := x_k + hg(x_k, x_{k-d}), \quad hd = 1,\ h\text{: step size.}$$

- Discrete Krisztin–Walther

$$x_{k+1} = \alpha x_k + \beta \arctan(\gamma x_{k-d}), \qquad \alpha \in (0,1),\ \beta > 0;$$

- Discrete Mackey–Glass I. (after translation)

$$x_{k+1} = \alpha x_k + \frac{\beta}{1 + x_{k-d}^p}, \qquad \alpha \in (0,1),\ \beta > 0,\ p > 1;$$

- Discrete Mackey–Glass II. (after translation)

$$x_{k+1} = \alpha x_k + \frac{\beta x_{k-d}}{1 + x_{k-d}^p}, \qquad \alpha \in (0,1),\ \beta > 0,\ p \in (0,1];$$

- May's genotype model

$$x_{k+1} = x_k + \beta \tanh(x_{k-d}/2), \qquad \beta \neq 0;$$

- Wazewska-Czyzewska and Lasota equation (after translation)

$$x_{k+1} = \alpha x_k + \beta e^{-\gamma x_{k-d}}, \qquad \alpha \in (0,1),\ \beta > 0,\ \gamma > 0.$$

- Ricker's equation (after translation)

$$x_{k+1} = x_k e^{\beta - x_{k-d}}, \qquad \beta > 0;$$

- Pielou's equation (after translation)

$$x_{k+1} = \frac{x_k}{\alpha + \beta x_{k-d}} \qquad \alpha \in (0,1),\ \beta > 0;$$

A continuous-time equation by J. Mallet-Paret, G. Sell, 1996

$$\begin{cases} \dot{x}_0(t) = f_0(t, x_0(t), x_1(t)) \\ \dot{x}_i(t) = f_i(t, x_{i-1}(t), x_i(t), x_{i+1}(t)) \quad \text{for } 1 \leq i \leq n-1, \\ \dot{x}_n(t) = f_n(t, x_{n-1}(t), x_n(t), x_0(t-1)) \end{cases} \tag{3}$$

assuming a positive feedback in variable x_{i-1} and either positive or negative feedback w.r.t. the last variable in f_i.

A continuous-time equation by J. Mallet-Paret, G. Sell, 1996

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assuming a positive feedback in variable x_{i-1} and either positive or negative feedback w.r.t. the last variable in f_i. Phase space: $C(\mathbb{K}) = C(\mathbb{K}, \mathbb{R})$, where $\mathbb{K} = [-1, 0] \cup \{1, 2, \ldots, n\}$.

Definition (Discrete Lyapunov functional)

Let $V_{\mathbb{K}} : C(\mathbb{K}) \setminus \{0\} \to \{0, 2, 4, \ldots, \infty\}$ be defined by

$$V_{\mathbb{K}}(\phi) = \begin{cases} \mathrm{sc}(\phi, \mathbb{K}), & \text{if } \mathrm{sc}(\phi, \mathbb{K}) \text{ is even or infinite,} \\ \mathrm{sc}(\phi, \mathbb{K}) + 1, & \text{if } \mathrm{sc}(\phi, \mathbb{K}) \text{ is odd.} \end{cases}$$

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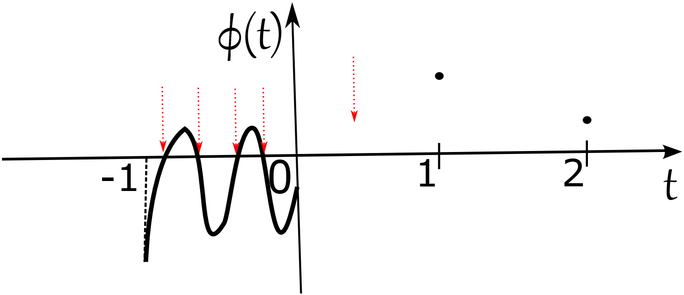


$$n = 2: \qquad \mathrm{sc}(\phi, \mathbb{K}) = 5 \quad \Rightarrow \quad V_{\mathbb{K}}(\phi) = 6$$

J. Mallet-Paret, 1988

- $f \in C^\infty$,
- negative feedback in 2nd variable,
- $D_2 f(0,0) < 0$, $D_1 f(0,0) + D_2 f(0,0) < 0$.

Morse sets $\approx$ level sets of a similarly defined discrete Lyapunov functional.

M. Polner, 2002

$$\dot{x}(t) = f(x(t), x(t-1))$$

- $f \in C^1$,
- positive feedback in the 2nd variable,
- $D_2 f(0,0) > 0$, $D_1 f(0,0) + D_2 f(0,0) > 0$

Morse sets $\approx$ level sets the discrete Lyapunov functional defined by Mallet-Paret (1996).

Euler-discretization of (3) by J. Mallet-Paret, G. Sell, 2003

$$\begin{cases} y_{k+1}(i) = y_k(i) + h\Big(B_k(i)((1-\theta)y_{k+1}(i) + \theta y_k(i)) \\ \qquad\qquad + A_k(i)y_k(i-1) + C_k(i)y_k(i+1)\Big) \\ y_{k+1}(j) = y_k(j+1), \qquad 0 \le i \le N, \quad -M \le j \le -1 \end{cases}$$

where $Mh = 1$, $\theta \in [0,1]$, $y_k(N+1) := \pm y_k(-M)$. The following assumptions are also required

- $A_k(i), B_k(i), C_k(i) \in \ell^\infty(\mathbb{N}_0, \mathbb{R})$ for all $0 \le i \le N$, $A_k(0) \equiv 0$;
- $A_k(i) \le 0$, $C_k(i) \ge 0$;
- $2h\|B(i)\|_\infty \le 1$, $5h\max\{\|A(i)\|_\infty, \|C(i)\|_\infty\} \le 1$.

Lyapunov f.: $\mathbb{R}^{M+N+1} \ni y \mapsto V(y) = \begin{cases} \mathrm{sc}(y), & \text{if } \mathrm{sc}(y) \text{ is even}, \\ \mathrm{sc}(y)+1, & \text{if } \mathrm{sc}(y) \text{ is odd}. \end{cases}$

M^*: # e.v. λ, s.t. $|\lambda| > 1$. $N^* := \begin{cases} M^*, & \text{if } 0 \text{ is hyp. or } M^* = 0, \\ M^* + 1, & \text{else.} \end{cases}$

Theorem

The following collection of sets is a Morse decomposition of the global attractor $\mathcal{A}$ of (2)

$$\mathcal{M}_n = \Big\{\xi \in \mathcal{A} \setminus \{0\} : \text{(2) has a bounded entire solution } (y_k)_{k\in\mathbb{Z}} \text{ through } \xi \text{ with } V^{\pm}(y_k) \equiv n \text{ on } \mathbb{Z} \text{ and } 0 \notin \alpha(y) \cup \omega(\xi)\Big\}$$

$$\mathcal{M}_{N^*} = \Big\{\xi \in \mathcal{A} \setminus \{0\} : \text{(2) has a bounded entire solution } (y_k)_{k\in\mathbb{Z}} \text{ through } \xi \text{ with } V^{\pm}(y_k) \equiv N^* \text{ on } \mathbb{Z}\Big\} \cup \{0\}$$

for $0 \le n \le d+1$, $n \in 2\mathbb{N}_0 \setminus \{N^*\}$

Regular set:

$$\mathcal{R}^+ := \left\{ y \in \mathbb{R}^{d+1} \;\middle|\; \begin{array}{l} y(i) = 0 \Rightarrow y(i-1)y(i+1) < 0 \text{ for } 0 \leq i \leq d \\ \text{with } y(d+1) := y(0) \text{ and } y(-1) := y(d) \end{array} \right\}.$$

Proposition

If $(y^n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathbb{R}^{d+1} \setminus \{0\}$ *with limit* $y \in \mathbb{R}^{d+1} \setminus \{0\}$*, then the following statements hold:*

(i) $V(y) \leq \liminf_{n\to\infty} V(y^n)$*;*

(ii) $V(y) = \lim_{n\to\infty} V(y^n)$*, if* $y \in \mathcal{R}^+$*.*

Furthermore, if y_k *is a solution of* (2) *for all* k *(on a discrete interval), then*

(iii) $V(y_{k+1}) \leq V(y_k)$ *for all* k*;*

(iv) if $y_k \neq 0$ *holds for some* $k \in \mathbb{Z}$ *and* $V(y_k) = V(y_{k+4d+1})$*, then* $y_{k+4d+1} \in \mathcal{R}^+$*.*

Characteristic equation $(a = D_1 f(0,0),\ b = D_2 f(0,0))$

$$\lambda^{d+1} - a\lambda^d - b = 0, \tag{4}$$

$$d^* := \begin{cases} \frac{d-1}{2}, & d \text{ is odd}, \\ \frac{d}{2}, & d \text{ is even}. \end{cases}$$

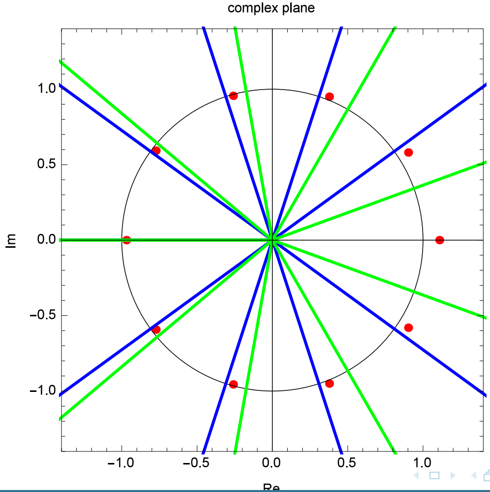
Lemma (G. Huszár, 1930)

Let $a, b > 0$. The characteristic roots $\lambda_j, \overline{\lambda_j}$, $j \in \{0, \ldots, d^*\}$, of (4) can be ordered according to

$$\lambda_j \in S_j := \left\{ z \in \mathbb{C} : \frac{(2j-1)\pi}{d} < \arg z < \frac{2j\pi}{d+1} \right\} \ \forall j \in \{1, \ldots, d^*\},$$

moreover, for even delays d there exists a unique real root of (4), namely $\lambda_0 > 0$ and $|\lambda_{d^*}| < \cdots < |\lambda_1| < \lambda_0$. For odd delays d there exist exactly two real roots of (4), namely $\lambda_{d^*+1} < 0 < \frac{ad}{d+1} < \lambda_0$ and $|\lambda_{d^*+1}| < |\lambda_{d^*}| < \cdots < |\lambda_1| < \lambda_0$.

Characteristic roots



Lemma

There exists an open neighbourhood U of 0, s.t. if y_k is a nontrivial solution of

$$y_{k+1} = F(y_k), \qquad F(y(0), \dots, y(d)) := \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ f(y(d), y(0)) \end{pmatrix} \tag{2}$$

such that

(i) if $y_k \in \overline{U} \cap \mathcal{A}$ for all $k \leq 0 \Rightarrow V(y_k) \leq N^*$ for all $k \in \mathbb{Z}$;

(ii) if $y_k \in \overline{U} \cap \mathcal{A}$ for all $k \geq 0 \Rightarrow V(y_k) \geq N^*$ for all $k \in \mathbb{Z}$.

Open problems, to-do-list

- Extend results to more general equations:
  - Properties of V (regularity?);
  - Morse-decomposition.
- Are all the Morse sets M_n nonempty for $n < N^*$ (so the ones with $V \leq N^*$)?

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Thank you!